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LV. The gudermannian complement and imaginary geometry

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The first group has a fair photographic action, the second a strong action. Taking into consideration the intensity of their photographic action and the variation of ionization with velocity, these two groups should contain a relatively large number of β rays. Consequently, a magnetic field was used which would direct all β rays of velocity less than $2{\cdot}8\times10^{10}\frac{\text{cm.}}{\text{sec.}}$ on to the plate P, and a balance obtained. Then the usual field was excited, and a balance again obtained. Some slight but rather uncertain evidence of the excitation of γ rays was obtained, but the decay of the emanation during the course of the observations made measurement of such small effects very difficult. Even had a constant source of β rays been available, the difference in amount of excited γ radiation was too small to measure with certainty.

If it be supposed that the γ rays from radioactive matter are produced by the β rays, these experiments bring out clearly the high efficiency of the transformation of β rays into γ rays during the disintegration of the atom of radium C compared with the efficiency of the conversion of β rays into γ rays when the former fall on matter of high atomic weight. As Gray has shown, the exact converse holds for the product radium E.

Similar experiments have been made, using emanation contained in a glass tube sufficiently thin to let out the α rays, and it has been found that a measurable amount of γ rays is apparently produced by the impact of α rays. The discussion of these experiments is reserved for a later paper.

I desire to express my best thanks to Prof. Rutherford for suggesting this research, and for his help and interest throughout the course of the experiments.

LV. *The Gudermannian Complement and Imaginary Geometry.* By George F. Becker*.

In applications of the gudermannian to physical problems, it is in many cases convenient to employ the complement of this angle rather than the angle itself. This slight modification also helps to bring out interesting analogies.

Let the gudermannian complement be denoted by

$$G(u)=\pi/2-\operatorname{gd} u\,;$$

then the familiar relations between the gudermannian and

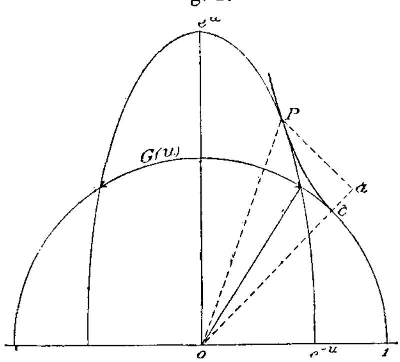
* Communicated by the Author.

hyperbolic functions become

$$\cot G(u) = \sinh u,$$
$$1/\sin G(u) = \cosh u,$$
$$\cos G(u) = \tanh u,$$
$$\tan\left[\frac{\pi}{4} - \tfrac{1}{2}G(u)\right] = \frac{1-\sin G(u)}{\cos G(u)} = \tanh\frac{u}{2},$$
$$\cot \tfrac{1}{2}G(u) = e^{u}.$$

The gudermannian complement is not a recondite function useful only on rare occasions, but one of the simplest and most useful of all functions. For instance, in an ellipse, let a mean proportional between the axes be selected as the unit of length, and about the centre of the ellipse let a unit circle be described as in fig. 1. The area of this circle and

Fig. 1.



Area $ocP \doteq u/2$; $oa \doteq \cosh u$; $aP \doteq \sinh u$.

of the ellipse will be equal, and for that reason it has been suggested that the diameters passing through the intersections of the two curves should be called isocyclic diameters *. The acute angle between these diameters is $G(u)$, the axes are e^{u} and e^{-u}, or $\cot \frac{1}{2}G(u)$ and $\tan \frac{1}{2}G(u)$; the tangent of the angle which either isocyclic diameter makes with the major axis is also e^{-u}, and the square of the focal distance is $2 \cot G(2u) = 2 \sinh (2u)$. In short, the gudermannian complement specifies the ellipse more succinctly than does either the eccentricity or the ellipticity, and ought to be introduced into the elementary geometry of that curve.

If the ellipse is regarded as derived by finite strain from

* Smithsonian Math. Tables, p. xxxi.

the circle whose radius is the geometric mean of the axes, this function has still other luminous properties. In a pure shear the particles which in the unstrained state lie along a radius at 45° to the axes lie after strain along the isocyclic diameter, this material line having described an angle $\pi/4-\frac{1}{2}\mathrm{G}(u)$. This same radius during strain sweeps over an area $u/2$ which is an hyperbolic sector. In scission (slide, shearing motion) the angle of rotation is $90°-\mathrm{G}(u)$, and one isocyclic diameter remains stationary while the other sweeps an area u. The amount of shear is $2\cot \mathrm{G}(u)$.

Since all deformations can be resolved into pure shears and scissions, all deformations can be reduced to terms of the gudermannian complement, and no simpler treatment has yet been suggested *.

Consider an ellipsoid of three distinct axes $h\alpha$, $h\beta$, $h\gamma$. This may be regarded as the strain ellipsoid derived from a sphere of radius $(\alpha\beta\gamma)^{1/3}$ by a dilatation of ratio h, and by distortion. It is evidently legitimate to take this radius as unity, so that $\alpha\beta\gamma=1$. The volume of the strain ellipsoid is then $\frac{4}{3}\pi h^3$, and for no strain $h=1$. Let $\alpha>\beta>\gamma$ and consider the ratios of the axes, $h\alpha/h\beta$, $h\beta/h\gamma$, $h\alpha/h\gamma$. These are all greater than unity if there is any strain at all. Hence they may be represented by hyperbolic cosines, thus

$$\frac{h\alpha}{h\beta}=\cosh a;\ \frac{h\beta}{h\alpha}=\cosh b;\ \frac{h\alpha}{h\gamma}=\cosh c,$$

and here $$\cosh a \,.\, \cosh b=\cosh c.$$

If the gudermannian is substituted in this equation,

$$\cos \mathrm{gd}\, a \cos \mathrm{gd}\, b=\cos \mathrm{gd}\, c,$$

showing at once that the three angles are those subtending the sides of a right-angled spherical triangle as shown in

Fig. 2.

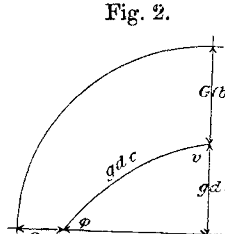


fig. 2. The angles opposite gd a and gd b are marked respectively υ and ϕ, but the complements of these angles are

* See Smithsonian Math. Tables, introduction; and Bull. Geol. Soc. Amer. vol. iv. 1893, p. 13.

more interesting than the angles themselves. Putting $A=90^\circ-\phi$ and $B=90^\circ-\upsilon$, it appears at once that

$$\tan A = \frac{\tanh a}{\sinh b} = \frac{\cos G(a)}{\cot G(b)},$$

$$\tan B = \frac{\tanh b}{\sinh a} = \frac{\cos G(b)}{\cot G(a)}.$$

Here it is known that A is the angle which the circular section of the strain ellipsoid under discussion makes with its greatest axis *, and it is evident that B may be considered as the angle made by the circular section of a second ellipsoid with its greatest axis. The axes of this second ellipsoid are $1/h\alpha$, $1/h\beta$, $1/h\gamma$; in other words, it is the reciprocal of the first. A and B are each in general less than $\pi/4$, a value which they reach only for vanishing strain.

A very close relation exists between a strain ellipsoid and its reciprocal. This is most readily seen in the case of a mass subjected only to finite, homogeneous deformation, which can always be represented by two shears of ratio α and β. If these two shears have their axes of extension in common they deform the sphere into an ellipsoid whose axes are $\alpha\beta$, $1/\alpha$, $1/\beta$. If the same two shears are combined by their contractile axes, they yield an ellipsoid whose axes are $1/\alpha\beta$, α, and β. This second ellipsoid is thus the reciprocal of the first. The loads (or initial stresses, or stresses into areas) are the same in absolute value in the two cases, but with signs reversed; so that equal finite loads of opposite signs produce deformations of reciprocal ratios.

A shear results from the action of two loads, Q and P, at right angles to one another if $P=-Q$. Q acting by itself would produce a dilatation of ratio say h_1, and P a cubical contraction of ratio say $1/h_2$. Acting together to produce a shear they must give a dilatation $h=h_1/h_2$. But since a shear is undilatational, $h=1$. Therefore equal forces of opposite signs produce not merely deformations but strains of reciprocal ratios.

Hence an elastic, homogeneous sphere subjected to a finite homogeneous strain and then allowed to vibrate is converted into the reciprocal ellipsoid at the opposite phase. This

* Smithsonian Math. Tables, p. xxxii.

vibration is not harmonic; it becomes harmonic, however, for infinitesimal amplitudes *.

Thus the angles A and B are of interest in the dynamics of a vibrating elastic mass.

Interesting also are the planes of maximum tangential strain or maximum slide. In the general ellipsoid there are four sets of such planes; two of these are symmetrically placed with reference to the greatest axis, make equal acute angles with it, and are perpendicular to the plane of the greatest and least axes. The other two are symmetrical with reference to the intermediate axis with which they make acute angles, and their directrices lie in the plane of this and the least axis. The strain along the first pair of planes is maximax and along the second pair minimax.

The angles which these planes make with the axes are dependent wholly on pure deformation, and are independent of dilatational stresses or pure rotation. They are thus reducible to terms of pure shear. Any number of pure axial

* The finite elastic load-strain function, whatever it is, must fulfil the reciprocal conditions set forth above. It is manifest that they would be satisfied by the hypothesis

$$\alpha = e^{Q/6n}, \quad h = e^{Q/9k}, \quad \alpha^2 h = e^{Q/M},$$

where n is the modulus of rigidity, k the modulus of cubical dilatation, and M Young's modulus. To the best of my knowledge, I long since proved that only this hypothesis will satisfy the conditions (Amer. Journ. Sci. vol. xlvi. 1893, p. 337). Finite strain, however, seems to excite almost no interest, and, so far as I know, the only authority who has discussed my conclusions is Ostwald, who approved them (*Zeitsch. Phys. Chemie*, vol. xiii. 1894, p. 136). These functions can, of course, be expressed in terms of the gudermannian complement, or, in other words, u may be regarded as a load expressed in terms of an appropriate modulus.

The hypothesis leads to the inference that Poisson's ratio

$$\sigma = \frac{-dx}{x} \Big/ \frac{dy}{y}$$

is constant, irrespective of the state of elastic strain. Here $x/x_0 = h/\alpha$ and $y/y_0 = \alpha^2 h$. By integration it follows that the equation of continuity is $xy^\sigma = x_0 y_0^\sigma$, which is unquestionably true for the three cases $\sigma = 1/2$, $\sigma = 0$, and $\sigma = -1$. Of these the first is nearly realized by indiarubber and the second by cork. The case of an infinitely rigid but compressible mass is given by $\sigma = -1$ or, what amounts to the same thing, the case of a real mass subjected only to hydrostatic pressure.

Only on this hypothesis will the frequency of vibrations be independent of their amplitude.

For small strains let $\alpha^2 h - 1 = f$, then

$$f = Q/M + \ldots.$$

which is Hooke's law, and my results do not conflict in any way with those of the classical investigations of elasticity, which, however, they tend to simplify.

shears of finite amount are reducible to two lying in planes at right angles to one another having one axis in common, and neither produces any relative tangential motion or slide in the plane of the other. In a pure shear of ratio $\alpha = e^u$ the planes of maximum tangential strain stand at an angle to the major axis of $\frac{1}{2}G(u) = \cot^{-1}\alpha$. For this case they are circular sections of the shear ellipsoid, and in them lie the isocyclic diameters. If a shear of ratio γ, at right angles to the plane of α, $1/\alpha$, is superposed on this strain, then the sheets of particles subject to maximum tangential strain by the first shear are deflected into a new position, and now make an angle ω with the axis α. It has been shown and is easily seen that

$$\tan\omega = \frac{1}{\alpha\gamma} = \beta.$$

If the γ-shear had been applied first there would have been maximum slide at $\cot^{-1}\gamma$, and this by the α-shear would be reduced to $\cot^{-1}(\gamma\,.\,\alpha)$. Thus each of the four sets of planes of maximum tangential strain makes with the least axis an angle $\pi/2-\omega$.

In finite extensions as well as in infinitesimal ones a tensile load Q is divisible into thirds, of which one produces dilatation and each of the others a shear. From two of Cauchy's stress quadrics it may be shown that the load on any central plane of the shear ellipsoid is the same or Q/3. On planes whose directrices are the principal axes this load is normal. There are two other planes on which it is wholly tangential, and these make with the greatest axis an angle whose cotangent is the ratio of shear. In other words, they are planes of maximum slide as well as of maximum tangential load. Maximum tangential stress, on the other hand, occurs at 45° to the axis of stress for any homogeneous strain, and on planes at this angle the slide falls short of a maximum.

Many substances, like mild steel, if ruptured by tension, part along surfaces which make angles with the axis of stress not very different from 45°, and most homogeneous substances ruptured by pressure break at similar angles to the axis of stress. The fragments into which rupture divides a specimen must show at least a little elastic recovery, which in the case of tensile loads would increase the apparent angle of planes of parting to the axis of stress, and in the case of rupture by pressure would decrease these angles. Hence, if rupture actually occurred on planes at 45° to the axis, the specimens broken by tension would show surfaces inclined to the direction of force by somewhat more than 45°, and specimens broken by pressure would have faces at less than 45° to the axis. As a matter of fact, rupture at sensibly 45° occurs only in brittle substances such as glass, which break when the deformation is extremely small. The scrap-heap of any testing shop shows that mild steel and similar materials under tension part on surfaces standing at less (instead of more) than 45° to the direction of stress, and that crushing produces partings at more (instead of less) than 45° to this direction. Prof. J. A. Ewing has summarized the experimental data as showing that the partings in crushed cast iron stand at about 55° to the axis of stress, and those in steel under tension at about 25° to this axis, pointing out that these substances distinctly do not yield on planes of maximum tangential stress *. Observations of my own on rocks, colloids, and pseudosolids as well as on metals show similar relations. The actual partings vary from 45° in the same sense as do the planes of maximum tangential strain, and vary with the plasticity of the material in the same sense as does the position of these planes.

Everything known to me is consistent with the theory that not merely the ruptures produced in most materials by testing-machines take place on planes of maximum slide or load, but also the joints, faults, and slaty cleavage of rocks †.

For geophysical purposes it is thus vital to find tan ω or β

* Enc. Brit. 11th ed. vol. xxv. p. 1016.

† Although indiarubber is a substance with very anomalous elastic properties, it seems to me that the behaviour of various preparations of rubber affords valuable suggestions. Thus an ordinary rubber band, made of raw rubber with some 10 per cent. of sulphur and vulcanized, may be stretched to about 8 times its length, but beyond this point is almost as inextensible as a bit of twine. For the buffers of railway carriages and similar purposes rubber is mixed with something like 35 per cent. of mineral matter for the express purpose of limiting the deformation of which it is capable. This addition changes the Young's modulus, as it evidently must, and it is also clear that a block of such rubber cannot yield to pressure elastically beyond the point at which the mineral particles are forced nearly into contact. Although the elastic limit is much reduced by the addition of mineral matter, the elasticity is not materially impaired; for such springs often do duty for a considerable number of years. Now a real elastic solid might be conceived as a mixture of particles almost incapable of deformation with a substance ideally elastic or capable of indefinite stretching and dilatation. Such a mixture would have a definite and possibly small elastic limit, but would be perfectly elastic within this limit. Above the elastic limit it must either break or yield through a permanent readjustment in the distribution of the contained inelastic particles which might be conceived as of molecular dimensions.

in terms of the other functions*. Now

$$\beta^3 = \frac{\beta^2}{\alpha\gamma} = \frac{\cosh b}{\cosh a} = \frac{\sin (2\mathrm{B})}{\sin (2\mathrm{A})} = \frac{\sin \mathrm{G}(a)}{\sin \mathrm{G}(b)},$$

and thus the angle B is a significant feature of the strain ellipsoid itself, as well as of the reciprocal ellipsoid.

It is noteworthy that in the simple case of pure deformation discussed the circular functions give the relative positions of points in the mass to one another, while the hyperbolic functions appertain to the lines of flow or to the absolute motions of the particles. This distinction is not preserved, however, in the case of rotational strains.

Having thus outlined the parts played in finite homogeneous strain, flow, and rupture of rocks and other solids by the gudermannian complement and the angles A and B, it may be well to write down for reference a few of the formulæ connecting these functions :—

$$\sin \mathrm{A} = \frac{\cot \mathrm{G}(a)}{\cot \mathrm{G}(c)} = \frac{\sinh a}{\sinh c}.$$

$$\cos \mathrm{A} = \frac{\cos \mathrm{G}(b)}{\cos \mathrm{G}(c)} = \frac{\tanh b}{\tanh c}.$$

$$\cos \mathrm{A} = \frac{\sin \mathrm{B}}{\sin \mathrm{G}(a)} = \sin \mathrm{B} \cosh a.$$

$$\cot \mathrm{A} = \frac{\cot \mathrm{G}(b)}{\cos \mathrm{G}(a)} = \frac{\sinh b}{\tanh a}.$$

$$\cos \mathrm{B} = \frac{\cos \mathrm{G}(a)}{\cos \mathrm{G}(c)} = \frac{\tanh a}{\tanh c}.$$

$$\cos \mathrm{B} = \frac{\sin \mathrm{A}}{\sin \mathrm{G}(b)} = \sin \mathrm{A} \cosh b.$$

$$\sin \mathrm{B} = \frac{\cot \mathrm{G}(b)}{\cot \mathrm{G}(c)} = \frac{\sinh b}{\sinh c}.$$

$$\cot \mathrm{B} = \frac{\cot \mathrm{G}(a)}{\cos \mathrm{G}(b)} = \frac{\sinh a}{\tanh b}.$$

$$\tan \mathrm{A} \text{ and } \mathrm{B} = \sin \mathrm{G}(c) = \sin \mathrm{G}(a) \sin \mathrm{G}(b).$$
$$= \operatorname{sech} c = \operatorname{sech} a \operatorname{sech} b.$$

$$\mathrm{A} + \mathrm{B} < \pi/2.$$

Many readers will perceive at a glance that these are

* Bull. Geol. Soc. America, vol. iv. 1893, p. 13; and Amer. Journ. Sci. vol. xxiv. 1907, p. 1.

neither more nor less than the standard formulæ of Lobachevsky's imaginary or hyperbolic geometry for right-angled triangles, the negative curvature being taken at unity. The only difference is that for $\Pi(u)$ I have used $G(u)$.

Whether space has negative curvature or not *, Lobachevsky's functions are applicable to physical relations so familiar as to be commonplace, and for this purpose they are more advantageous than any others yet devised.

In physical questions the gudermannian complement should be employed for the most part rather than the gudermannian, and these functions deserve to be introduced into elementary mathematics †.

Since Lobachevsky's $\Pi(u)$ is equivalent to the gudermannian complement, a table of the gudermannian such as that in 'Smithsonian Mathematical Tables' becomes a table of $\Pi(u)$ by the simple process of subtracting the tabulated values from $\pi/2$, and this opens the way to the employment of these functions in fields other than that of geology, notably in electricity.

Washington, D.C.,
May 24, 1912.

LVI. *On Potential Measurements in the Neighbourhood of the Electrodes in Point-plane Discharge.* By Miss P. M. Borthwick, *B.Sc.* ‡

IN a paper on the ionizing processes at the surface of a point discharging in air (Phil. Mag. Aug. 1910), it was shown by Drs. Chattock and Tyndall that the value of the field at the surface of a discharging point, under various conditions, could be obtained by measuring the pull of the lines of force on its end, which was made hemispherical for the purpose. These field measurements were made by them, and by Dr. Tyndall, for varying values of current, distance, and size of point, both when the point was opposite a plate and when it was subjected to a stream of ions arriving at its end from an external source. These ions were of opposite sign to that of the point and are referred to below as "external ions."

* If the entity in which the properties of æther inhere is ponderable, the æther must be confined to the portions of space characterized by stellar bodies, the æther must be in a state of strain, and the path of light is not rectilinear for Euclidian space.

† For elementary work "deformation angle" might perhaps be a better term than gudermannian complement.

‡ Communicated by Dr. A. M. Tyndall.