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METHOD OF SYMMETRICAL CO-ORDINATES APPLIED TO THE SOLUTION OF POLYPHASE NETWORKS

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ABSTRACT OF PAPER

In the introduction a general discussion of unsymmetrical systems of co-planar vectors leads to the conclusion that they may be represented by symmetrical systems of the same number of vectors, the number of symmetrical systems required to define the given system being equal to its degrees of freedom. A few trigonometrical theorems which are to be used in the paper are called to mind. The paper is subdivided into three parts, an abstract of which follows. It is recommended that only that part of Part I up to formula (33) and the portion dealing with star-delta transformations be read before proceeding with Part II.

Part I deals with the resolution of unsymmetrical groups of numbers into symmetrical groups. These numbers may represent rotating vectors of systems of operators. A new operator termed the sequence operator is introduced which simplifies the manipulation. Formulas are derived for three-phase circuits. Star-delta transformations for symmetrical co-ordinates are given and expressions for power deduced. A short discussion of harmonics in three-phase systems is given.

Part II deals with the practical application of this method to symmetrical rotating machines operating on unsymmetrical circuits. General formulas are derived and such special cases, as the single-phase induction motor, synchronous motor-generator, phase converters of various types, are discussed.

INTRODUCTION

IN THE latter part of 1913 the writer had occasion to investigate mathematically the operation of induction motors under unbalanced conditions. The work was first carried out, having particularly in mind the determination of the operating characteristics of phase converters which may be considered as a particular case of unbalanced motor operation, but the scope of the subject broadened out very quickly and the writer undertook this paper in the belief that the subject would be of interest to many.

The most striking thing about the results obtained was their

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symmetry; the solution always reduced to the sum of two or more symmetrical solutions. The writer was then led to inquire if there were no general principles by which the solution of unbalanced polyphase systems could be reduced to the solution of two or more balanced cases. The present paper is an endeavor to present a general method of solving polyphase network which has peculiar advantages when applied to the type of polyphase networks which include rotating machines.

In physical investigations success depends often on a happy choice of co-ordinates. An electrical network being a dynamic system should also be aided by the selection of a suitable system of co-ordinates. The co-ordinates of a system are quantities which when given, completely define the system. Thus a system of three co-planar congruent vectors are defined when their magnitude and their angular position with respect to some fixed direction are given. Such a system may be said to have six degrees of freedom, for each vector may vary in magnitude and phase position without regard to the others. If, however, we impose the condition that the vector sum of these vectors shall be zero, we find that with the direction of one vector given, the other two vectors are completely defined when their magnitude alone is given, the system has therefore lost two degrees of freedom by imposing the above condition which in dynamical theory is termed a "constraint". If we impose a further condition that the vectors be symmetrically disposed about their common origin this system will now have but two degrees of freedom.

It is evident from the above definition that a system of n coplanar congruent vectors may have $2n$ degrees of freedom and that a system of n symmetrically spaced vectors of equal magnitude has but two degrees of freedom. It should be possible then by a simple transformation to define the system of n arbitrary congruent vectors by n other systems of congruent vectors which are symmetrical and have a common point. The n symmetrical systems so obtained are the symmetrical co-ordinates of the given system of vectors and completely define it.

This method of representing polyphase systems has been employed in the past to a limited extent, but up to the present time there has been as far as the author is aware no systematic presentation of the method. The writer hopes by this paper to interest others in the application of the method, which will be found to be a valuable instrument for the solution of certain classes of polyphase networks.

In dealing with alternating currents in this paper, use is made of the complex variable which in its most general form may be represented as a vector of variable length rotating about a given point at variable angular velocity or better as the resultant of a number of vectors each of constant length rotating at different angular velocities in the same direction about a given point. This vector is represented in the text by $\check{I}$, $\check{E}$, etc., and the conjugate vector which rotates at the same speed in the opposite direction is represented by $\hat{I}$, $\hat{E}$, etc. The effective value of the vector is represented by the symbol without the distinguishing mark as I, E, etc. The impedances Z_a, Z_b, Z_{ab}, etc., are generally functions of the operator, $D = \dfrac{d}{dt}$ and the characteristics of the circuit; these characteristics are constants only when there is no physical motion. It will therefore be necessary to carefully distinguish between $Z_a\, \check{I}_a$ and $\check{I}_a\, Z_a$ when Z_a has the form of a differential operator. In the first case a differential operation is carried out on the time variable $\check{I}_a$ in the second case the differential operator is merely multiplied by $\check{I}_a$.

The most general expression for a simple harmonic quantity e is

$$e = A \cos pt - B \sin pt$$

in exponential form this becomes

$$e = \frac{A + j\,B}{2}\, e^{j\,pt} + \frac{A - j\,B}{2} e^{-j\,pt}$$

$(A + j\,B)\, e^{j\,pt}$ represents a vector of length $\sqrt{A^2 + B^2}$ rotating in the positive direction with angular velocity p while $(A - j\,B)$ $e^{-j\,pt}$ is the conjugate vector rotating at the same angular velocity in the opposite direction. Since $e^{j\,pt}$ is equal to $\cos pt + j \sin pt$, the positively rotating vector $\check{E} = (A + j\,B)\, e^{j\,pt}$ will be

$$\check{E} = A \cos pt - B \sin pt + j\,(A \sin pt + B \cos pt)$$

or the real part of $\check{E}$ which is its projection on a given axis is equal to e and therefore $\check{E}$ may be taken to represent e in phase and magnitude. It should be noted that the conjugate vector $\hat{E}$ is equally available, but it is not so convenient since the operation $\frac{d}{dt} e^{-j\,pt}$ gives $-j\,p\,e^{-j\,pt}$ and the imaginary part of the impedance operator will have a negative sign.

The complex roots of unity will be referred to from time to time in the paper. Thus the complete solution of the equation $x^n - 1 = 0$ requires n different values of x, only one of which is real when n is an odd integer. To obtain the other roots we have the relation

$$\begin{aligned} 1 &= \cos 2\,\pi\,r + j \sin 2\,\pi\,r \\ &= e^{j\,2\,\pi\,r} \end{aligned}$$

Where r is any integer. We have therefore

$$1^{\frac{1}{n}} = e^{j\frac{2\,\pi\,r}{n}}$$

and by giving successive integral values to r from 1 to n, all the n roots of $X^n - 1 = 0$ are obtained namely,

$$\begin{aligned} a_1 &= e^{j\frac{2\,\pi}{n}} = \cos \frac{2\,\pi}{n} + j \sin \frac{2\,\pi}{n} \\ a_2 &= e^{j\frac{4\,\pi}{n}} \quad \cos \frac{4\,\pi}{n} + j \sin \frac{4\,\pi}{n} \\ a_3 &= e^{j\frac{6\,\pi}{n}} \quad \cos \frac{6\,\pi}{n} + j \sin \frac{6\,\pi}{n} \\ a_n &= e^{j\,2\,\pi} = 1 \end{aligned}$$

It will be observed that $a_2\; a_3 \ldots . a_n$ are respectively equal to $a_1^2\; a_1^3 \ldots . a_1^{(n-1)}$.

When there is relative motion between the different parts of a circuit as for example in rotating machinery, the mutual inductances enter into the equation as time variables and when the motion is angular the quantities e^{jwt} and e^{-jwt} will appear in the operators. In this case we do not reject the portion of the operator having e^{-jwt} as a factor, because the equations require that each vector shall be operated on by the operator as a whole which when it takes the form of a harmonic time function will contain terms with e^{jwt} and e^{-jwt} in conjugate relation. In some cases as a result of this, solutions will appear with indices of e which are negative time variables; in such cases the vectors with negative index should be replaced by their conjugates which rotate in the positive direction.

This paper is subdivided as follows:

Part I.—"The Method of Symmetrical Co-ordinates." Deals with the theory of the method, and its application to simple polyphase circuits.

Part II.—Application to Symmetrical Machines on Unbalanced Polyphase Circuits. Takes up Induction Motors, Generator and Synchronous Motor, Phase Balancers and Phase Convertors.

Part III. Application to Machines having Unsymmetrical Windings.

In the Appendix the mathematical representation of field forms and the derivation of the constants of different forms of networks is taken up.

The portions of Part I dealing with unsymmetrical windings are not required for the applications taken up in Part II and may be deferred in a later reading. The greater part of Part I is taken up in deriving formulas for special cases from the general formulae (**30**) and (**33**), and the reading of the text following these equations may be confined to the special cases of immediate interest.

I wish to express my appreciation of the valuable help and suggestions that have been given me in the preparation of this paper by Prof. Karapetoff who suggested that the subject be presented in a mathematical paper and by Dr. J. Slepian to whom I am indebted for the idea of sequence operators and by others who have been interested in the paper.

PART I

Method of Symmetrical Generalized Co-ordinates

Resolution of Unbalanced Systems of Vectors and Operators

The complex time function $\check{E}$ may be used instead of the harmonic time function e in any equation algebraic or differential in which it appears linearly. The reason of this is because if any linear operation is performed on $\check{E}$ the same operation performed on its conjugate $\hat{E}$ will give a result which is conjugate to that obtained from $\check{E}$, and the sum of the two results obtained is a solution of the same operation performed on $\check{E} + \hat{E}$, or $2\,e$.

It is customary to interpret $\check{E}$ and $\hat{E}$ as coplanar vectors, rotating about a common point and e as the projection of either vector on a given line, $\check{E}$ being a positively rotating vector and

$\hat{E}$ being a negatively rotating vector, and their projection on the given line being

$$e = \frac{\check{E} + \hat{E}}{2} \tag{1}$$

Obviously if this interpretation is accepted one of the two vectors becomes superfluous and the positively rotating vector $\check{E}$ may be taken to represent the variable "e" and we may define "e" by saying that "e" is the projection of the vector $\check{E}$ on a given line or else by saying that "e" is the real part of the complex variable $\check{E}$.

If $1, a, a^2 \ldots . a^{n-1}$ are the n roots of the equation $x^n - 1 = 0$ a symmetrical polyphase system of n phases may be represented by

$$\left.\begin{aligned} \check{E}_{11} &= \check{E}_{11} \\ \check{E}_{21} &= a\, \check{E}_{11} \\ \check{E}_{31} &= a^2\, \check{E}_{11} \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \check{E}_{n1} &= a^{n-1}\, \check{E}_{11} \end{aligned}\right\} \tag{2}$$

Another n phase system may be obtained by taking

$$\left.\begin{aligned} \check{E}_{12} &= \check{E}_{12} \\ \check{E}_{22} &= a^2\, \check{E}_{12} \\ \check{E}_{32} &= a^4\, \check{E}_{12} \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \check{E}_{n2} &= a^{2\,(n-1)} \check{E}_{12} \end{aligned}\right\} \tag{3}$$

and this also is symmetrical, although it is entirely different from (**2**).

Since $1 + a + a^2 + a^{n-1} = 0$, the sum of all the vectors of a symmetrical polyphase system is zero.

If $\check{E}_1\ \check{E}_2\ \check{E}_3 \ldots . \check{E}_n$ be a system of n vectors, the following identities may be proved by inspection:

$$
\left.
\begin{aligned}
\check{E}_1 \equiv {} & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots . \check{E}_n}{n} \\
& + \frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots . a^{n-1}\,\check{E}_n}{n} \\
& + \frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\check{E}_3 + \ldots . a^{2(n-1)}\,\check{E}_n}{n} \\
& + \frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + a^{2(r-1)}\,\check{E}_3 + \ldots . a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + \ldots . \frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots . a^{-(n-1)}\,\check{E}_n}{n} \\
\check{E}_2 \equiv {} & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots . \check{E}_n}{n} \\
& + a^{-1}\,\frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots . a^{n-1}\,\check{E}_n}{n} \\
& + a^{-2}\,\frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\,\check{E}_3 + \ldots . a^{2(n-1)}\,\check{E}_n}{n} \\
& + a^{-(r-1)}\,\frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + a^{2(r-1)}\,\check{E}_3 + a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + a^{-(n-1)}\,\frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots . a^{-(n-1)}\,\check{E}_n}{n} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\check{E}_n \equiv {} & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots . \check{E}_n}{n} \\
& + a^{-(n-1)}\,\frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots . a^{n-1}\,\check{E}_n}{n} \\
& + a^{-2(n-1)}\,\frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\,\check{E}_4 + \ldots . a^{2(n-1)}\,\check{E}_n}{n} \\
& + a^{-(n-1)(r-1)}\,\frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + \ldots . a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + a^{-1}\,\frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots . a^{-(n-1)}\,\check{E}_n}{n}
\end{aligned}
\right\} \quad (4)
$$

It will be noted that in the expression for $\check{E}_1$ in the above formulae if the first term of each component is taken the result is

$n \frac{\check{E}_1}{n}$ or $\check{E}_1$. If the succeeding terms of each component involving $\check{E}_2 \check{E}_3 \ldots \check{E}_n$ respectively, are taken separately they add up to expressions of the form $\frac{\check{E}_r}{n}$ $(1+a+a^2+\ldots a^{n-1})$ which are all equal to zero since $(1+a+a^2+\ldots a^{n-1})$ is equal to zero. In like manner in the expression for $\check{E}_2 \check{E}_3 \ldots \check{E}_n$ respectively, all the terms of the components involving each of the quantities $\check{E}_1 \check{E}_2 \check{E}_3 \ldots$ etc. excepting the terms involving that one of which the components are to be determined add up to expressions of the form $\frac{\check{E}_r}{n}$ $(1 + a + a^2 + \ldots a^{n-1})$ all of which are equal to zero, the remaining terms add up to $\check{E}_2 \check{E}_3 \ldots E_n$ respectively. It will now be apparent that (**4**), is true whatever may be the nature of $\check{E}_1 \check{E}_2$ etc., and therefore it is true of all numbers, real complex or imaginary, whatever they may represent and therefore similar relations may be obtained for current vectors and they may be extended to include not only vectors but also the operators.

In order to simplify the expressions which become unwieldy when applied to the general n phase system, let us consider a three phase system of vectors $\check{E}_a \check{E}_b \check{E}_c$. Then we have the following identities:

$$\left.\begin{aligned}
\check{E}_a &\equiv \frac{\check{E}_a + \check{E}_b + \check{E}_c}{3} + \frac{E_a + a\,\check{E}_b + a^2\,E_c}{3} \\
&\qquad + \frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3} \\
\check{E}_b &\equiv \frac{\check{E}_a + \check{E}_b + \check{E}_c}{3} + a^2\,\frac{\check{E}_a + a\,\check{E}_b + a^2\,E_c}{3} \\
&\qquad + a\,\frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3} \\
\check{E}_c &\equiv \frac{\check{E}_a + \check{E}_b + \check{E}_c}{3} + a\,\frac{\check{E}_a + a\,\check{E}_b + a^2\,\check{E}_c}{3} \\
&\qquad + a^2\,\frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3}
\end{aligned}\right\} \quad (\mathbf{5})$$

(**4**) states the law that a system of n vectors or quantities may be resolved when n is prime into n different symmetrical groups or systems, one of which consists of n equal vectors and the remaining $(n - 1)$ systems consist of n equispaced vectors which with the first mentioned groups of equal vectors forms an equal number of symmetrical n-phase systems. When n is not prime some of the n-phase systems degenerate into repetitions of systems having numbers of phases corresponding to the factors of n.

Equation (**5**) states that any three vectors $\check{E}_a$ $\check{E}_b$ $\check{E}_c$ may be

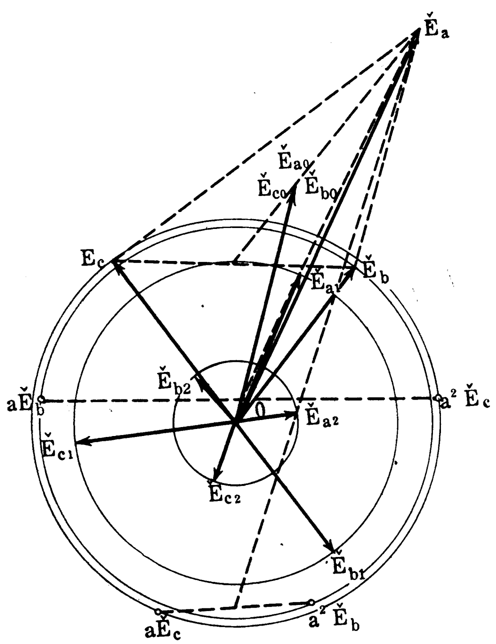

FIG. 1—GRAPHICAL REPRESENTATION OF EQUATION 5.

resolved into a system of three equal vectors $\check{E}_{a0}$ $\check{E}_{a0}$ $\check{E}_{a0}$ and two symmetrical three phase systems $\check{E}_{a1}$, a^2 $\check{E}_{a1}$, a $\check{E}_{a1}$, $\check{E}_{a2}$, a $\check{E}_{a2}$, a^2 $\check{E}_{a2}$, the first of which is of positive phase sequence and the second of negative phase sequence, or

$$\left.\begin{aligned} \check{E}_a &= \check{E}_{a0} + \check{E}_{a1} + \check{E}_{a2} \\ \check{E}_b &= \check{E}_{a0} + a^2\,\check{E}_{a1} + a\,\check{E}_{a2} \\ \check{E}_c &= \check{E}_{a0} + a\,\check{E}_{a1} + a^2\,\check{E}_{a2} \end{aligned}\right\} \quad (6)$$

Similarly

$$\left.\begin{aligned}\check{I}_a &= \check{I}_{a0} + \check{I}_{a1} + \check{I}_{a2}\\ \check{I}_b &= \check{I}_{a0} + a_2\, \check{I}_{a1} + a\, \check{I}_{a2}\\ \check{I}_c &= \check{I}_{a0} + a\, \check{I}_{a1} + a^2\, \check{I}_{a2}\end{aligned}\right\} \qquad (\mathbf{7})$$

Figs. (1) and (2) show a graphical method of resolving three vectors into their symmetrical three-phase components corresponding to equations (**5**).

The system of operators $Z_{aa}\ Z_{bb}\ Z_{cc}\ Z_{ab}\ Z_{bc}\ Z_{ca}$ may be resolved in a similar manner into symmetrical groups,

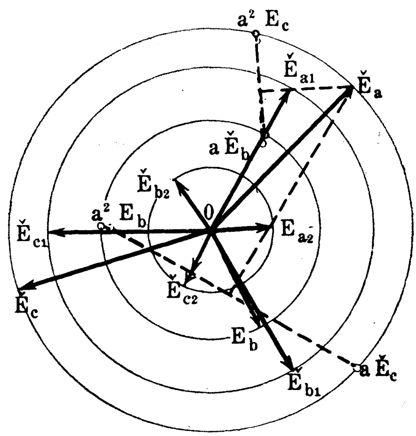

FIG. 2—GRAPHICAL REPRESENTATION OF EQUATION 5.

$$\left.\begin{aligned}Z_{aa} &= Z_{aa0} + Z_{aa1} + Z_{aa2}\\ Z_{bb} &= Z_{aa0} + a^2\, Z_{aa1} + a\, Z_{aa2}\\ Z_{cc} &= Z_{aa0} + a\, Z_{aa1} + a^2\, Z_{aa2}\end{aligned}\right\} \qquad (\mathbf{8})$$

$$\left.\begin{aligned}Z_{ab} &= Z_{ab0} + Z_{ab1} + Z_{ab2}\\ Z_{bc} &= Z_{ab0} + a^2\, Z_{ab1} + a\, Z_{ab2}\\ Z_{ca} &= Z_{ab0} + a\, Z_{ab1} + a^2\, Z_{ab2}\end{aligned}\right\} \qquad (\mathbf{9})$$

There are similar relations for n phase systems.

Explanation of Theory and Use of Sequence Operator

Consider the following sequences of nth roots of unity:

$$\left.\begin{array}{l} S^0 = 1, \quad 1, \quad 1 \ldots . 1 \\ S^1 = 1, \quad a^{-1}, \quad a^{-2} \ldots . a^{-(n-1)} \\ S^2 = 1, \quad a^{-2}, \quad a^{-4} \ldots . a^{-2(n-1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ S^r = 1, \quad a^{-r}, \quad a^{-2r} \ldots . a^{-(n-1)r} \\ S^{r+1} = 1, \quad a^{-(r+1)}, \quad a^{-2(r+1)} \ldots . a^{-(n-1)(r+1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ S^{n-1} = 1, \quad a^{-(n-1)}, \quad a^{-2(n-1)} \ldots . a^{-(n-1)^2} \end{array}\right\} \quad (10)$$

Consider the sequence obtained by the products of similar terms of S^r and S^1. It will be

$$S^{r+1} = 1, \quad a^{-(r+1)}, \quad a^{-2(r+1)} \ldots . a^{-(n-1)(r+1)} \quad (11)$$

Similarly

$$S^k = 1, \quad a^{-k}, \quad a^{-2k} \ldots . a^{-(n-1)k} \quad (12)$$

and the sequence obtained by products of like terms of this sequence and S^r is

$$S^{r+k} = 1, \quad a^{-(r+k)}, \quad a^{-2(r+k)} \ldots . a^{-(n-1)(r+k)} \quad (13)$$

We may therefore apply the law of indices to the products of sequences to obtain the resulting sequence.

In the case of the three-phase system we shall have the following sequences only to consider, viz.:

$$\left.\begin{array}{l} S^0 = 1, \quad 1, \quad 1 \\ S^1 = 1, \quad a^2, \quad a \\ S^2 = 1, \quad a, \quad a^2 \end{array}\right\} \quad (14)$$

The complete system of currents $\check{I}_a \; \check{I}_b \; \check{I}_c$ are defined by

$$S\,(\check{I}_a) = S^0 \, \check{I}_{a0} + S^1 \, \check{I}_{a1} + S^2 \, \check{I}_{a2} \quad (15)$$

Similarly the impedances $Z_{aa} \; Z_{bb} \; Z_{cc}$ may be expressed in symmetrical form

$$S\,(Z_{aa}) \equiv S^0 \, Z_{aa0} + S^1 \, Z_{aa1} + S^2 \, Z_{aa2} \quad (16)$$

and the mutual impedances Z_{ab}, Z_{bc}, Z_{ca} are expressed by

$$S\,(Z_{ab}) \equiv S^0 \, Z_{ab0} + S^1 \, Z_{ab1} + S^2 \, Z_{ab2} \quad (17)$$

Attention is called to the importance of preserving the cyclic order of self and mutual impedances, otherwise the rule for the sequence operator will not hold. Thus, Z_{ab}, Z_{bc} and Z_{ca} are in proper sequence as also are Z_{ca}, Z_{ab}, Z_{bc}.

When it is desired to change the first term in the sequence of polyphase vectors the resulting expression will be

$$\left.\begin{aligned} S\,(\check{I}_b) &= S^0\,\check{I}_{a0} + S^1\,a^2\,\check{I}_{a1} + S^2\,a\,\check{I}_{a2} \\ S\,(\check{I}_c) &= S^0\,\check{I}_{a0} + S^1\,a\,\check{I}_{a1} + S^2\,a^2\,\check{I}_{a2} \end{aligned}\right\} \qquad (18)$$

Similarly in the case of the operators $S\,(Z_{ab})$ we have

$$\left.\begin{aligned} S\,(Z_{bc}) &= S^0\,Z_{ab0} + S^1\,a^2\,Z_{ab1} + S^2\,a\,Z_{ab2} \\ S\,(Z_{ca}) &= S^0\,Z_{ab0} + S^1\,a\,Z_{ab1} + S^2\,a^2\,Z_{ab2} \end{aligned}\right\} \qquad (19)$$

Similar rules apply to the e.m.fs. $E_a\ E_b\ E_c$

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^0\,\check{E}_{a0} + S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2} \\ S\,(\check{E}_b) &= S^0\,\check{E}_{a0} + S^1\,a^2\,\check{E}_{a1} + S^2\,a\,\check{E}_{a2} \\ S\,(\check{E}_c) &= S^0\,\check{E}_{a0} + S^1\,a\,E_{a1} + S^2\,a^2\,\check{E}_{a2} \end{aligned}\right\} \qquad (20)$$

It should be kept in mind that any one of the several expressions $S\,(\check{I}_a)\ S\,(\check{I}_b)\ S\,(\check{I}_c)$, etc., completely specifies the system, and each of the members of the groups of equations given above is a complete statement of the system of vectors or operators and their relation.

Application to Self and Mutual Impedance Operations

We may now proceed with the current, systems $S\,(\check{I}_a)$, $S\,(\check{I}_b)$, $S\,(\check{I}_c)$ and the operating groups $S\,(Z_{aa})\ S\,(Z_{bb})\ S\,(Z_{cc})$ etc. and the electromotive forces in exactly the same manner as for simple a-c. circuits. Thus,

$$\begin{aligned} S\,(\check{E}_a) &= S\,(Z_{aa})\,S\,(\check{I}_a) + S\,(Z_{ab})\,S\,(\check{I}_b) + S\,(Z_{ca})\,S\,(\check{I}_c) \qquad (21) \\ &= (S^0\,Z_{aa0} + S^1\,Z_{aa1} + S^2\,Z_{aa2})\,(S^0\,\check{I}_{a0} + S^1\,\check{I}_{a1} + S^2\,I_{a2}) \\ &\quad + (S^0\,Z_{ab0} + S^1\,Z_{ab1} + S^2\,Z_{ab2}) \\ &\qquad\qquad (S^0\,\check{I}_{a0} + S^1\,a^2\,\check{I}_a + S^2\,a\,\check{I}_{a2}) \\ &\quad + (S^0\,Z_{ab0} + S^1\,a\,Z_{ab1} + S^2\,a^2\,Z_{ab2}) \\ &\qquad\qquad (S^0\,\check{I}_{a0} + S^1\,a\,\check{I}_{a1} + S^2\,a^2\,\check{I}_{a2}) \\ &= S^0\,(Z_{aa0} + 2\,Z_{ab0})\,\check{I}_{a0} + S^0\,\{Z_{aa2} + (1 + a^2)\,Z_{ab2}\}\,\check{I}_{a1} \end{aligned}$$

$$+ S^0 \{Z_{aa1} + (1 + a) Z_{ab1}\} \check{I}_{a2}$$
$$+ S^1 \{Z_{aa1} + (1 + a) Z_{ab1}\} \check{I}_{a0}$$
$$+ S^1 \{Z_{aa0} + (a + a^2) Z_{ab0}\} \check{I}_{a1}$$
$$+ S^1 \{Z_{aa2} + 2\, a\, Z_{ab2}\} \check{I}_{a2}$$
$$+ S^2 \{Z_{aa2} + (1 + a^2) Z_{ab2}\} \check{I}_{a0}$$
$$+ S^2 \{Z_{aa1} + 2\, a^2 Z_{ab1}\} \check{I}_{a1}$$
$$+ S^2 \{Z_{aa0} + (a + a^2) Z_{ab0}\} \check{I}_{a2} \qquad \textbf{(22)}$$

Or since $1 + a + a^2 = 0$, $1 + a = - a^2$, $1 + a^2 = - a$ and $a + a^2 = - 1$

$$S(\check{E}_a) = S^0 (Z_{aa0} + 2\, Z_{ab0}) \check{I}_{a0} + S^0 (Z_{aa2} - a\, Z_{ab2}) \check{I}_{a1}$$
$$+ S^0 (Z_{aa1} - a^2 Z_{ab1}) \check{I}_{a2} + S^1 (Z_{aa1} - a^2 Z_{ab1}) \check{I}_{a0}$$
$$+ S^1 (Z_{aa0} - Z_{ab0}) \check{I}_{a1} + S^1 (Z_{aa2} + 2\, a\, Z_{ab2}) \check{I}_{a2}$$
$$+ S^2 (Z_{aa2} - a\, Z_{ab2}) \check{I}_{a0} + S^2 (Z_{aa1} + 2\, a^2 Z_{ab1}) \check{I}_{a1}$$
$$+ S^2 (Z_{aa0} - Z_{ab0}) \check{I}_{a2} \qquad \textbf{(23)}$$

Or since

$$S(Z_{bc}) = S^0 Z_{bc0} + S^1 Z_{bc1} + S^2 Z_{bc2}$$
$$= S^0 Z_{ab0} + S^1 a^2 Z_{ab1} + S^2 a\, Z_{ab2}$$

we may write (**23**) in the form

$$S(\check{E}_a) = S^0 (Z_{aa0} + 2\, Z_{bc0}) \check{I}_{a0} + S^0 (Z_{aa2} - Z_{bc2}) \check{I}_{a1}$$
$$+ S^0 (Z_{aa1} - Z_{bc1}) \check{I}_{a2} + S^1 (Z_{aa1} - Z_{bc1}) \check{I}_{a0}$$
$$+ S^1 (Z_{aa0} - Z_{bc0}) \check{I}_{a1} + S^1 (Z_{aa2} + 2\, Z_{bc2}) \check{I}_{a2}$$
$$+ S^2 (Z_{aa2} - Z_{bc2}) \check{I}_{a0} + S^2 (Z_{aa1} + 2\, Z_{bc1}) \check{I}_{a1}$$
$$+ S^2 (Z_{aa0} - Z_{bc0}) \check{I}_{a2} \qquad \textbf{(24)}$$

which is the more symmetrical form. We have therefore from (**24**) by expressing $S(\check{E}_a)$ in terms of symmetrical co-ordinates the three symmetrical equations

$$\left.\begin{aligned}
S^0 \check{E}_{a0} &= S^0 \{(Z_{aa0} + 2\, Z_{bc0}) \check{I}_{a0} + (Z_{aa2} - Z_{bc2}) \check{I}_{a1} \\
&\qquad + (Z_{aa1} - Z_{bc1}) \check{I}_{a2}\} \\
S^1 E_{a1} &= S^1 \{(Z_{aa1} - Z_{bc1}) \check{I}_{a0} + (Z_{aa0} - Z_{bc0}) \check{I}_{a1} \\
&\qquad + (Z_{aa2} + 2\, Z_{bc2}) \check{I}_{a2}\} \\
S^2 E_{a2} &= S^2 \{(Z_{aa2} - Z_{bc2}) \check{I}_{a0} + (Z_{aa1} + 2\, Z_{bc1}) \check{I}_{a1} \\
&\qquad + (Z_{aa0} - Z_{bc0}) \check{I}_{a2}\}
\end{aligned}\right\} \qquad \textbf{(25)}$$

An important case to which we must next give consideration is that of mutual inductance between a primary polyphase circuit and a secondary polyphase circuit. The mutual impedances may be arranged in three sets. Let the currents in the secondary windings be I_u I_v and I_w, we may then express the generalized mutual impedances as follows:

$$\left.\begin{array}{r}\text{(I) } Z_{au}\ Z_{bv}\ Z_{cw}\\ \text{(II) } Z_{bw}\ Z_{cu}\ Z_{av}\\ \text{(III) } Z_{cv}\ Z_{aw}\ Z_{bu}\end{array}\right\}\qquad(26)$$

Each set may be resolved into three symmetrical groups, so that

$$\left.\begin{array}{l}S\ (Z_{au}) = S^0\ Z_{au0} + S^1\ Z_{au1} + S^2\ Z_{au2}\\ S\ (Z_{bw}) = S^0\ Z_{bw0} + S^1\ Z_{bw1} + S^2\ Z_{bw2}\\ S\ (Z_{cv}) = S^0\ Z_{cv0} + S^1\ Z_{cv1} + S^2\ Z_{cv2}\end{array}\right\}\qquad(27)$$

and we have for $S\ (\check{E}_a)$ the primary induced e.m.f. due to the secondary currents $S\ (\check{I}_u)$

$$S\ (\check{E}_a) = S\ (Z_{au})\ S\ (\check{I}_u) + S\ (Z_{av})\ S\ (\check{I}_v) + S\ (Z_{aw})\ S\ (\check{I}_w)\qquad(28)$$

Substituting for $S\ (\check{I}_u)$, $S\ (\check{I}_v)$ and $S\ (\check{I}_w)$ and $S\ (Z_{au})$, $S\ (Z_{av})$, $S\ (Z_{aw})$ their symmetrical equivalents we have

$$\begin{aligned}S\ (\check{E}_a) = {} & S^0\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ I_{uo}\\ & + S^0\ (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{u1}\\ & + S^0\ (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{u2}\\ & + S^1\ (Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{u0}\\ & + S^1\ (Z_{au0} + a^2\ Z_{bw0} + a\ Z_{cv0})\ \check{I}_{u1}\\ & + S^1\ (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{u2}\\ & + S^2\ (Z_{au2} + a^2\ Z_{bw2} + a\ Z_{v2})\ \check{I}_{u0}\\ & + S^2\ (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{u1}\\ & + S^2\ (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{u2}\end{aligned}\qquad(29)$$

On expressing $S\ (\check{E}_a)$ in symmetrical form we have the following three symmetrical equations

$$\left.\begin{aligned}
S^0\ \check{E}_{a0} &= S^0\ \{(Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{u0} \\
&\quad + (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{u1} \\
&\quad + (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{u2}\} \\
S^1\ \check{E}_{a1} &= S^1\ \{(Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{u0} \\
&\quad + (Z_{au0} + a^2\ Z_{bw0} + a\ Z_{cv0})\ \check{I}_{u1} \\
&\quad + (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{u2}\} \\
S^2\ E_{a2} &= S^2\ \{(Z_{au2} + a^2\ Z_{bw2} + a\ Z_{cv2})\ \check{I}_u{}^0 \\
&\quad + (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{u1} \\
&\quad + (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{u2}\}
\end{aligned}\right\} \quad \textbf{(30)}$$

For the e.m.f. $S\ (\check{E}_u)$ induced in the secondary by the primary currents $S\ (\check{I}_a)$ we have

$$S\ (\check{E}_u) = S\ (Z_{au})\ S\ (\check{I}_a) + S\ (Z_{bu})\ S\ (\check{I}_b) + S\ (Z_{cu})\ S\ (\check{I}_c) \quad \textbf{(31)}$$

Since $S\ (Z_{bu})$ bears the same relation to $S\ (Z_{cv})$ as $S\ (Z_{av})$ does to $S\ (Z_{bw})$ and $S\ (Z_{au})$ bears the same relation to $S\ (Z_{bw})$ as $S\ (Z_{aw})$ does to $S\ (Z_{cv})$ to obtain $S\ (\check{E}_u)$ all that will be necessary will be to interchange Z_{bw} and Z_{cv} in (**29**) and change $\check{I}_{u0}\ \check{I}_{u1}\ \check{I}_{u2}$ to $\check{I}_{a0}\ I_{a1}$ and $\check{I}_{a2}$ respectively, this gives

$$\begin{aligned}
S\ (\check{E}_u) &= S^0\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{a0} \\
&\quad + S^0\ (Z_{au2} + a^2\ Z_{bw2} + a\ Z_{cv2})\ \check{I}_{a1} \\
&\quad + S^0\ (Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{a2} \\
&\quad + S^1\ (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{a0} \\
&\quad + S^1\ (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{a1} \\
&\quad + S^1\ (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{a2} \\
&\quad + S^2\ (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{a0} \\
&\quad + S^2\ (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{a1} \\
&\quad + S^2\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{a2}
\end{aligned} \quad \textbf{(32)}$$

and the three symmetrical equations will be

$$\left.\begin{aligned}
S^0\, \check{E}_{u0} &= S^0 \{(Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{a0} \\
&\quad + (Z_{au2} + a^2\, Z_{bw2} + a\, Z_{bw2} + a\, Z_{cv2})\, \check{I}_{a1} \\
&\quad + (Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1})\, \check{I}_{a2}\} \\
S^1\, \check{E}_{a1} &= S^1 \{(Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1})\, \check{I}_{a0} \\
&\quad + (Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0})\, \check{I}_{a1} \\
&\quad + (Z_{au2} + Z_{bw2} + Z_{cv2})\, \check{I}_{a2}\} \\
S^2\, \check{E}_{u2} &= S^2 \{(Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2})\, \check{I}_{a0} \\
&\quad + (Z_{au1} + Z_{bw1} + Z_{cv1})\, \check{I}_{a1} \\
&\quad + (Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0})\, \check{I}_{a2}\}
\end{aligned}\right\} \qquad (33)$$

The same methods may be applied to polyphase systems of any number of phase. When the number of phases is not prime the system may sometimes be dealt with as a number of polyphase systems having mutual inductance between them:—For example, a nine-phase system may be treated as three three-phase systems, a twelve phase system as three four-phase or four three-phase systems. In certain froms of dissymmetry this method is of great practical value, and its application will be taken up later.

For the present part of the paper we shall confine ourselves to the three-phase system, and dissymmetries of several different kinds.

The operators $Z_{au}\, Z_{aa}$, etc., must be interpreted in the broadest sense. They may be simple complex quantities or they may be functions of the differential operator $\frac{d}{dt}$. For if

$$i = \Sigma\, (A_n \cos n\, w\, t + B_n \sin n\, w\, t)$$

it may be expressed in the form

$$\left.\begin{aligned}
i &= \Sigma \left(\frac{A_n - j\, B_n}{2}\, e^{jnwt} + \frac{A_n + j\, B_n}{2}\, e^{-jnwt}\right) \\
&= \frac{\check{I}}{2} + \frac{\hat{I}}{2} \\
&= \text{real part of } \check{I}
\end{aligned}\right\} \qquad (34)$$

and any linear algebraic operation performed on $\check{I}/2$ will give a result which will be conjugate to that obtained by carrying out the same operation on $\hat{I}/2$ and since the true solution is the sum of these results, it may also be obtained by taking the real part of the result of performing the operation on $\check{I}$.

Modification of the General Case Met With in Practical Networks

Several symmetrical arrangements of the operator Z_{au} etc.' are frequently met with in practical networks which result in a much simpler system of equations than those obtained for the general case as in equations (**29**) to (**33**). Thus for example if all the operators in (**26**) are equal, all the operators in (**27**), except $S^0\, Z_{au0}$ $S^0\, Z_{bw0}$ and $S^0\, Z_{cv0}$ are equal to zero, and these three quantities are also equal to one another so that equation (**30**) becomes

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{u0} \\ S^1\, \check{E}_{a1} &= 0 \\ S^2\, E_{a2} &= 0 \end{aligned}\right\} \qquad (\mathbf{35})$$

and equation (**33**)

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{a0} \\ S^1\, E_{u1} &= 0 \\ S^2\, E_{u2} &= 0 \end{aligned}\right\} \qquad (\mathbf{36})$$

This is the statement in symmetrical co-ordinates that a symmetrically disposed polyphase transmission line will produce no electromagnetic induction in a second similar polyphase system so disposed with respect to the first that mutual inductions between all phases of the two are equal except that due to single-phase currents passing through the conductors.

If in (**26**) the quantities in each group only are equal, equations (**30**) and (**33**) become

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{u0} \\ S^1\, \check{E}_{a1} &= S^1\, (Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0})\, \check{I}_{u1} \\ S^2\, \check{E}_{a2} &= S^2\, (Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0})\, \check{I}_{u2} \end{aligned}\right\} \qquad (\mathbf{37})$$

$$\left.\begin{aligned} S^0\,\check{E}_{u0} &= S^0\,(Z_{au0} + Z_{bw0} + Z_{cv0})\,\check{I}_{a0} \\ S^1\,\check{E}_{a1} &= S^1\,(Z_{au0} + a\,Z_{bw0} + a^2\,Z_{cv0})\,\check{I}_{a1} \\ S^2\,\check{E}_{a2} &= S^1\,(Z_{au0} + a^2\,Z_{bw0} + a\,Z_{cv0})\,\check{I}_{a2} \end{aligned}\right\} \quad (38)$$

Symmetrical Forms of Common Occurrence

A symmetrical form which is of importance because it is of frequent occurrence in practical polyphase networks has the terms in group (I) equation (**26**) all equal and those in group (II) $\cos \frac{2\pi}{3}$ times those in group (I) and those in group (III) $\cos \frac{4\pi}{3}$ times those in group (I).

Since $\cos \frac{2\pi}{3} = \frac{a + a^2}{2} = \cos \frac{4\pi}{3}$ we have on substituting the values of the impedances in this case,

$$\left.\begin{aligned} S^0\,\check{E}_{a0} &= S^0\,\{Z_{au0}\,(1 + a + a^2)\}\,\check{I}_{u0} = 0 \\ S^1\,\check{E}_{a1} &= S^1\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{u1} \\ S^2\,E_{a2} &= S^2\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{u2} \end{aligned}\right\} \quad (39)$$

$$\left.\begin{aligned} S^0\,E_{u0} &= S^0\,\{Z_{au0}\,(1 + a + a^2)\}\,\check{I}_{a0} = 0 \\ S^1\,\check{E}_{u1} &= S^1\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{a1} \\ S^2\,\check{E}_{u2} &= S^2\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{a2} \end{aligned}\right\} \quad (40)$$

The elements in group I may be unequal but groups II and III may be obtained from group I by multiplying by $\cos \frac{4\pi}{3}$ and $\cos \frac{2\pi}{3}$ respectively.

The members of the three groups will then be related as follows, the same sequence being used as before,

$$\left.\begin{aligned} &(\mathrm{I}) \quad Z_{au}, \quad Z_{bv}, \quad Z_{cw} \\ &(\mathrm{II}) \quad \frac{a + a^2}{2}\,Z_{cw}, \quad \frac{a + a^2}{2}\,Z_{au}, \quad \frac{a + a^2}{2}\,Z_{bv} \\ &(\mathrm{III}) \quad \frac{a + a^2}{2}\,Z_{bv}, \quad \frac{a + a^2}{2}\,Z_{cw}, \quad \frac{a + a^2}{2}\,Z_{au} \end{aligned}\right\} \quad (41)$$

Consequently the following relations are true:

$$\left.\begin{aligned}
S^0\, Z_{bw0} &= \frac{a + a^2}{2}\, S^0\, Z_{au0} \\
S^0\, Z_{cv0} &= \frac{a + a^2}{2}\, S^0\, Z_{au0} \\
S^1\, Z_{bw1} &= \frac{1 + a^2}{2}\, S^1\, Z_{au1} \\
S^2\, Z_{bw2} &= \frac{1 + a}{2}\, S^2\, Z_{au2} \\
S^1\, Z_{cv1} &= \frac{1 + a}{2}\, S^1\, Z_{au1} \\
S^2\, Z_{cv2} &= \frac{1 + a^2}{2}\, S^2\, Z_{au2}
\end{aligned}\right\} \qquad (\mathbf{42})$$

Substituting these relations in (**30**) and (**33**) we have for this system of mutual impedances

$$\left.\begin{aligned}
Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\
Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{cv0} \\
Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{aw0}
\end{aligned}\right\} \qquad (\mathbf{43})$$

$$\left.\begin{aligned}
Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\
Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\
Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 0
\end{aligned}\right\} \qquad (\mathbf{44})$$

$$\left.\begin{aligned}
Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\
Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 0 \\
Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}
\end{aligned}\right\} \qquad (\mathbf{45})$$

which on substitution in (**30**) and (**33**) gives

$$\left.\begin{aligned}
S^0\, \check{E}_{a0} &= 0 \\
S^1\, \check{E}_{a1} &= S^1 \left\{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\right\} \\
S^2\, \check{E}_{a2} &= S^2 \left\{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2}\right\}
\end{aligned}\right\} \qquad (\mathbf{46})$$

$$\left.\begin{aligned}
S^0\, \check{E}_{u0} &= S^0 \left\{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a2}\right\} \\
S^1\, \check{E}_{u1} &= S^1 \left\{1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a2}\right\} \\
S^2\, \check{E}_{u2} &= S^2 \left\{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{a2}\right\}
\end{aligned}\right\} \qquad (\mathbf{47})$$

The above symmetrical forms in which the factors $\cos \frac{2\pi}{3}$ and $\cos \frac{4\pi}{3}$ occur apply particularly to electromagnetic induction between windings distributed over the surfaces of co-axial cylinders; where if the plane of symmetry of one winding be taken as the datum plane, the mutual impedance between this winding and any other is a harmonic function of the angle between its plane of symmetry and the datum plane. In other words, the mutual impedances are functions of position on the circumference of a circle and may therefore be expanded by Fourier's theorem in a series of integral harmonics of the angle made by the planes of symmetry with the datum plane. Since the same procedure applies to all the terms of the expansion it is necessary only to consider the simple harmonic case. In the partially symmetrical cases of mutual induction, such as that taken up in the preceding discussion, there will be a difference between two possible cases, viz:—Symmetrical primary, unsymmetrical secondary, which is the case just considered, and unsymmetrical primary and symmetrical secondary in which the impedances of (**26**) will have the following values

$$\left.\begin{array}{llll} \text{(I)} & Z_{au}, & Z_{bc}, & Z_{cw} \\ \text{(II)} & \frac{a + a^2}{2} Z_{bc}, & \frac{a + a^2}{2} Z_{cw}, & \frac{a + a^2}{2} Z_{au} \\ \text{(III)} & \frac{a + a^2}{2} Z_{cw}, & \frac{a + a^2}{2} Z_{au}, & \frac{a + a^2}{2} Z_{bc} \end{array}\right\} \quad (\mathbf{48})$$

The results may be immediately set down by symmetry from equations (**46**) and (**47**), but the difference between the two cases will be better appreciated by setting down the component symmetrical impedances, thus we have

$$\left.\begin{aligned} S^0 Z_{bw0} &= \frac{a + a^2}{2} S^0 Z_{au0} \\ S^0 Z_{cv0} &= \frac{a + a^2}{2} S^0 Z_{au0} \\ S^1 Z_{bw1} &= \frac{1 + a}{2} S^1 Z_{au1} \\ S^2 Z_{bw2} &= \frac{1 + a^2}{2} S^2 Z_{au2} \\ S^1 Z_{cv1} &= \frac{1 + a^2}{2} S^1 Z_{cu1} \\ S^2 Z_{cv2} &= \frac{1 + a}{2} S^2 Z_{au2} \end{aligned}\right\} \quad (\mathbf{49})$$

Substituting these relations in the impedances used in (**30**) and (**33**) they become

$$\left.\begin{aligned} Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\ Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0} \\ Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0} \end{aligned}\right\} \quad (\mathbf{50})$$

$$\left.\begin{aligned} Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\ Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 0 \\ Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \end{aligned}\right\} \quad (\mathbf{51})$$

$$\left.\begin{aligned} Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\ Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\ Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 0 \end{aligned}\right\} \quad (\mathbf{52})$$

And we have from (**30**) and (**33**), or by symmetry

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, \{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u2}\} \\ S^1\, \check{E}_{a1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{a2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2} \end{aligned}\right\} \quad (\mathbf{53})$$

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= 0 \\ S^1\, \check{E}_{u1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{u2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2}\} \end{aligned}\right\} \quad (\mathbf{54})$$

If the angle between the planes of symmetry of the coils and the datum plane are subject to changes, $\cos \frac{2\pi}{3}$ and $\cos \frac{4\pi}{3}$ in the preceding discussion must be replaced by

$$\left.\begin{aligned} \cos\left(\frac{2\pi}{3} + \theta\right) &= \frac{a}{2}\, e^{j\theta} + \frac{a^2}{2}\, e^{-j\theta} \\ \cos\left(\frac{4\pi}{3} + \theta\right) &= \frac{a}{2}\, e^{-j\theta} + \frac{a^2}{2}\, e^{j\theta} \end{aligned}\right\} \quad (\mathbf{55})$$

where θ is measured from the datum plane

In the strictly symmetrical case of co-axial cylindrical surface windings in which the members of each group of mutual impedances are equal, the result of substituting (**55**) in the equations for induced e.m.f. will be

$$\left.\begin{aligned} S^0\,\check{E}_{a0} &= 0 \\ S^1\,\check{E}_{a1} &= S^1\,(1\tfrac{1}{2}\,Z_{au0}\,e^{j\theta}\;\check{I}_{u1}) \\ S^2\,\check{E}_{a2} &= S^2\,(1\tfrac{1}{2}\,Z_{au0}\,e^{-j\theta}\;\check{I}_{u2}) \end{aligned}\right\} \qquad (56)$$

$$\left.\begin{aligned} S^0\,\check{E}_{u0} &= 0 \\ S^1\,\check{E}_{u1} &= S^1\,(1\tfrac{1}{2}\,Z_{au0}\,e^{-j\theta}\;\check{I}_{a1}) \\ S^2\,\check{E}_{u2} &= S^2\,(1\tfrac{1}{2}\,Z_{au0}\,e^{j\theta}\;\check{I}_{a2}) \end{aligned}\right\} \qquad (57)$$

In the case having symmetrical primary and unsymmetrical secondary in which members of each group are different, but in which there are harmonic relations between corresponding members of the different groups, the impedances are

$$\left.\begin{aligned} &\text{(I)}\; Z_{au},\quad Z_{bv},\quad Z_{cw} \\ &\text{(II)}\left(\frac{a}{2}\,e^{j\theta} + \frac{a^2}{2}\,e^{-j\theta}\right) Z_{cw} \\ &\left(\frac{a}{2}\,e^{j\theta} + \frac{a^2}{2}\,e^{-j\theta}\right) Z_{au},\; \left(\frac{a}{2}\,e^{j\theta} + \frac{a^2}{2}\,e^{-j\theta}\right) Z_{bv} \\ &\text{(III)}\left(\frac{a^2}{2}\,e^{j\theta} + \frac{a}{2}\,e^{-j\theta}\right) Z_{bv}, \\ &\left(\frac{a^2}{2}\,e^{j\theta} + \frac{a}{2}\,e^{-j\theta}\right) Z_{cw},\; \left(\frac{a^2}{2}\,e^{j\theta} + \frac{a}{2}\,e^{-j\theta}\right) Z_{au} \end{aligned}\right\} \qquad (58)$$

The symmetrical component mutual impedances will have the following values in terms of $Z_{au0}\; Z_{au1}\; Z_{au2}$

$$\left.\begin{aligned} S^0\,Z_{bw0} &= \left(\frac{a}{2}\,e^{j\theta} + \frac{a^2}{2}\,e^{-j\theta}\right) S^0\,Z_{au0} \\ S^0\,Z_{cv0} &= \left(\frac{a^2}{2}\,e^{j\theta} + \frac{a}{2}\,e^{-j\theta}\right) S^0\,Z_{au0} \\ S^1\,Z_{bw1} &= \left(\frac{a^2}{2}\,e^{j\theta} + \frac{e^{-j\theta}}{2}\right) S^1\,Z_{au1} \\ S^2\,Z_{bw2} &= \left(\frac{e^{j\theta}}{2} + \frac{a}{2}\,e^{-j\theta}\right) S^2\,Z_{au2} \\ S^1\,Z_{cv1} &= \left(\frac{a}{2}\,e^{j\theta} + \frac{e^{-j\theta}}{2}\right) S^1\,Z_{au1} \\ S^2\,Z_{cv2} &= \left(\frac{e^{j\theta}}{2} + \frac{a^2}{2}\,e^{-j\theta}\right) S^2\,Z_{au2} \end{aligned}\right\} \qquad (59)$$

Substituting these relations in the impedances of equations (**30**) and (**33**) they become

$$\left.\begin{aligned} Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\ Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, e^{-j\theta} \\ Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, e^{j\theta} \end{aligned}\right\} \qquad (\mathbf{60})$$

$$\left.\begin{aligned} Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, e^{-j\theta} \\ Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, e^{j\theta} \\ Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 0 \end{aligned}\right\} \qquad (\mathbf{61})$$

$$\left.\begin{aligned} Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}\, e^{j\theta} \\ Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 0 \\ Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au}\, e^{-j\theta} \end{aligned}\right\} \qquad (\mathbf{62})$$

which on substitution in (**30**) and (**33**) give

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= 0 \\ S^1\, \check{E}_{a1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au1}\, e^{j\theta}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au0}\, e^{j\theta}\, \check{I}_{u1} \\ &\qquad + 1\tfrac{1}{2}\, Z_{au2}\, e^{j\theta}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{a2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au2}\, e^{-j\theta}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au1}\, e^{-j\theta}\, \check{I}_{u1} \\ &\qquad + 1\tfrac{1}{2}\, Z_{au0}\, e^{-j\theta}\, \check{I}_{u2}\} \end{aligned}\right\} \qquad (\mathbf{63})$$

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= S^0\, \{1\tfrac{1}{2}\, Z_{au2}\, e^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au1}\, e^{j\theta}\, \check{I}_{a2}\} \\ S^1\, \check{E}_{u1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au0}\, e^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au2}\, e^{j\theta}\, \check{I}_{a2}\} \\ S^2\, \check{E}_{u2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au1}\, e^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au0}\, e^{j\theta}\, \check{I}_{a2}\} \end{aligned}\right\} \qquad (\mathbf{64})$$

In the case of unsymmetrical primary and symmetrical secondary, we have for the value of the impedance in terms of Z_{au0} Z_{au1} and Z_{au2}

$$\left.\begin{aligned} &(\mathrm{I})\ Z_{au},\quad Z_{bv},\quad Z_{cw} \\ &(\mathrm{II}) \left(\frac{a}{2}\, e^{j\theta} + \frac{a^2}{2}\, e^{-j\theta}\right) Z_{bc}, \\ &\left(\frac{a}{2}\, e^{j\theta} + \frac{a^2}{2}\, e^{-j\theta}\right) Z_{cw},\ \left(\frac{a}{2}\, e^{j\theta} + \frac{a^2}{2}\, e^{-j\theta}\right) Z_{au} \\ &(\mathrm{III}) \left(\frac{a^2}{2}\, e^{j\theta} + \frac{a}{2}\, e^{-j\theta}\right) Z_{cw}, \\ &\left(\frac{a^2}{2}\, e^{j\theta} + \frac{a}{2}\, e^{-j\theta}\right) Z_{au},\ \left(\frac{a^2}{2}\, e^{j\theta} + \frac{a}{2}\, e^{-j\theta}\right) Z_{bc} \end{aligned}\right\} \qquad (\mathbf{65})$$

The symmetrical component mutual impedances in terms of Z_{au0}, Z_{au1}, Z_{au2} are

$$\left.\begin{aligned}
S^0 Z_{bw0} &= \left(\frac{a}{2} e^{j\theta} + \frac{a^2}{2} e^{-j\theta}\right) S^0 Z_{au0} \\
S^0 Z_{cv0} &= \left(\frac{a^2}{2} e^{j\theta} + \frac{a}{2} e^{-j\theta}\right) S^0 Z_{au0} \\
S^1 Z_{bw1} &= \left(\frac{e^{j\theta}}{2} + \frac{a}{2} e^{-j\theta}\right) S^1 Z_{au1} \\
S^2 Z_{bw2} &= \left(\frac{a^2}{2} e^{j\theta} + \frac{e^{-j\theta}}{2}\right) S^2 Z_{au2} \\
S^1 Z_{cv1} &= \left(\frac{e^{j\theta}}{2} + \frac{a^2}{2} e^{-j\theta}\right) S^1 Z_{au1} \\
S^2 Z_{cv2} &= \left(\frac{a}{2} e^{j\theta} + \frac{e^{-j\theta}}{2}\right) S^2 Z_{au2}
\end{aligned}\right\} \quad (66)$$

And the impedances of equations (**30**) and (**33**) become

$$\left.\begin{aligned}
Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\
Z_{au0} + a\, Z_{bw0} + a^2 Z_{cv0} &= 1\tfrac{1}{2} Z_{au0}\, e^{-j\theta} \\
Z_{au0} + a^2 Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2} Z_{au0}\, e^{j\theta}
\end{aligned}\right\} \quad (67)$$

$$\left.\begin{aligned}
Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2} Z_{au1}\, e^{j\theta} \\
Z_{au1} + a\, Z_{bw1} + a^2 Z_{cv1} &= 0 \\
Z_{au1} + a^2 Z_{bw1} + a\, Z_{cv1} &= 1\tfrac{1}{2} Z_{au1}\, e^{-j\theta}
\end{aligned}\right\} \quad (68)$$

$$\left.\begin{aligned}
Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2} Z_{au2}\, e^{-j\theta} \\
Z_{au2} + a\, Z_{bw2} + a^2 Z_{cv2} &= 1\tfrac{1}{2} Z_{au2}\, e^{j\theta} \\
Z_{au2} + a^2 Z_{bw} + a\, Z_{cv2} &= 0
\end{aligned}\right\} \quad (69)$$

And on substitution in (**30**) and (**33**), or by symmetry from (**63**) and (**64**), we have

$$\left.\begin{aligned}
S^0 \check{E}_{a0} &= S^0 \left\{1\tfrac{1}{2} Z_{au2}\, e^{j\theta}\ \check{I}_{u1} + 1\tfrac{1}{2} Z_{au1}\, e^{-j\theta}\ \check{I}_{u2}\right\} \\
S^1 \check{E}_{a1} &= S^1 \left\{1\tfrac{1}{2} Z_{au0}\, e^{j\theta}\ \check{I}_{u1} + 1\tfrac{1}{2} Z_{au2}\, e^{-j\theta}\ \check{I}_{u2}\right\} \\
S^2 \check{E}_{a2} &= S^2 \left\{1\tfrac{1}{2} Z_{au1}\, e^{j\theta}\ \check{I}_{u1} + 1\tfrac{1}{2} Z_{au0}\, e^{-j\theta}\ \check{I}_{u2}\right.
\end{aligned}\right\} \quad (70)$$

$$\left.\begin{aligned}
S^0\, \check{E}_{u0} &= 0 \\
S^1\, \check{E}_{u1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au1}\, e^{-j\theta}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au0}\, e^{-j\theta}\, \check{I}_{u1} \\
&\qquad + 1\tfrac{1}{2}\, Z_{au2}\, e^{-j\theta}\, \check{I}_{a2}\} \\
S^2\, \check{E}_{u2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au2}\, e^{j\theta}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au1}\, e^{j\theta}\, \check{I}_{a1} \\
&\qquad + 1\tfrac{1}{2}\, Z_{au0}\, e^{j\theta}\, \check{I}_{a2}\}
\end{aligned}\right\} \quad (71)$$

A fuller discussion of self and mutual impedances of co-axial cylindrical windings will be found in the Appendix. It will be sufficient to note here that in the case of self inductance and mutual inductance of stationary windings symmetrically disposed if they are equal

$$\left.\begin{aligned}
M_{ab} = M_{bc} = M_{ca} &= \Sigma\left(A_n \cos\frac{2\,n\,\pi}{3}\right) \\
L_{aa} = L_{bb} = L_{cc} = M_{aa} = M_{bb} = M_{cc} &= \Sigma\, A_n
\end{aligned}\right\} \quad (72)$$

If the windings are symmetrically disposed but have different number of turns

$$\left.\begin{aligned}
L_{aa} &= M_{aa} = \Sigma\, A_n \\
L_{bb} &= M_{bb} = \Sigma\, B_n \\
L_{cc} &= M_{cc} = \Sigma\, C_n
\end{aligned}\right\} \quad (73)$$

$$\left.\begin{aligned}
M_{ab} &= \Sigma\left(\sqrt{A_n\, B_n}\cos\frac{2\,n\,\pi}{3}\right) \\
M_{bc} &= \Sigma\left(\sqrt{B_n\, C_n}\cos\frac{2\,n\,\pi}{3}\right) \\
M_{ca} &= \Sigma\left(\sqrt{C_n\, A_n}\cos\frac{2\,n\,\pi}{3}\right)
\end{aligned}\right\} \quad (74)$$

If the coils are alike but unsymmetrically spaced $L_{aa}\, L_{bb}\, L_{cc}$ have the same values, namely $\Sigma\, A_n$ and

$$\left.\begin{aligned}
M_{ab} &= \Sigma\left\{(A_n \cos n\,\theta_1)\cos\frac{2\,n\,\pi}{3} + (A_n \sin n\,\theta_1)\sin\frac{2\,n\,\pi}{3}\right\} \\
M_{bc} &= \Sigma\left\{(A_n \cos n\,\theta_2)\cos\frac{2\,n\,\pi}{3} + (A_n \sin n\,\theta_2)\sin\frac{2\,n\,\pi}{3}\right\} \\
M_{ca} &= \Sigma\left\{(A_n \cos n\,\theta_3)\cos\frac{2\,n\,\pi}{3} + (A_n \sin n\,\theta_3)\sin\frac{2\,n\,\pi}{3}\right\}
\end{aligned}\right\} \quad (75)$$

If they are unequal as well as unsymmetrically disposed but are otherwise similar L_{aa} L_{bb} L_{cc} have values as in (**64**) and

$$\left.\begin{aligned}
M_{ab} &= \Sigma \left\{ (\sqrt{A_n B_n} \cos n\, \theta_1) \cos \frac{2\,n\,\pi}{3} \right. \\
&\qquad \left. + (\sqrt{A_n B_n} \sin n\, \theta_1) \sin \frac{2\,n\,\pi}{3} \right\} \\
M_{bc} &= \Sigma \left\{ (\sqrt{B_n C_n} \cos n\, \theta_2) \cos \frac{2\,n\,\pi}{3} \right. \\
&\qquad \left. + (\sqrt{B_n C_n} \sin n\, \theta_2) \sin \frac{2\,n\,\pi}{3} \right\} \\
M_{ca} &= \Sigma \left\{ (\sqrt{C_n A_n} \cos n\, \theta_1) \cos \frac{2\,n\,\pi}{3} \right. \\
&\qquad \left. + (\sqrt{C_n A_n} \sin n\, \theta_3) \sin \frac{2\,n\,\pi}{3} \right\}
\end{aligned}\right\} \qquad (\mathbf{76})$$

Where the windings are dissimilar in every respect the expressions become more complicated. A short outline of this subject is given in the Appendix.

In the case of mutual inductance between two coaxial cylindrical systems, one of which A, B, C is the primary and the other U, V, W the secondary, the following conventions should be followed:

(a) All angles are measured, taking the primary planes of symmetry as data in a positive direction.

(b) The datum plane for all windings is the plane of symmetry of the primary A phase.

(c) All mechanical motions unless otherwise stated shall be considered as positive rotations of the secondary cylinder about its axis.

(d) The conventional disposition of the phases and the direction of rotation of the secondary winding are indicated in Fig. 3.

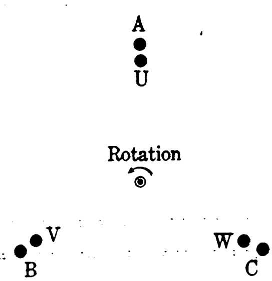

FIG. 3—CONVENTIONAL DISPOSITION OF PHASES AND DIRECTION OF ROTATION.

We shall consider five cases; Case 1 being the completely symmetrical case and the rest being symmetrical in one winding, the other winding being unsymmetrical in magnitude and phase, or both, but all windings having the same form and distribution of coils.

Case I. All Windings Symmetrical.

$$\left.\begin{aligned} M_{au} &= M_{bv} = M_{cw} = \Sigma A_n \cos n\,\theta \\ M_{bw} &= M_{cu} = M_{av} = \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= M_{aw} = M_{bu} = \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \qquad (77)$$

Case II. Primary Windings equal and Symmetrical, Secondary Windings unequal but otherwise Symmetrical.

$$\left.\begin{aligned} M_{au} &= \Sigma A_n \cos n\,\theta,\ M_{bv} = \Sigma B_n \cos n\,\theta,\ M_{cw} \\ &= \Sigma C_u \cos n\,\theta \\ M_{bw} &= \Sigma C_u \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{cu} &= \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{av} &= \Sigma B_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= \Sigma B_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{aw} &= \Sigma C_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{bu} &= \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \qquad (78)$$

Case III. Primary Winding Unequal but Otherwise Symmetrical, Secondary Winding &qual and Symmetrical.

$$\left.\begin{aligned} M_{au} &= \Sigma A_n \cos n\,\theta, M_{bv} = \Sigma B_n \cos n\,\theta,\ M_{cw} = \Sigma C_n \cos n\theta \\ M_{bw} &= \Sigma B_n \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{cu} &= \Sigma C_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{av} &= \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= \Sigma C_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{aw} &= \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{bu} &= \Sigma B_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \qquad (79)$$

Case IV. Same as Case II except in addition to inequality Secondary Windings are Displaced from Symmetry by angles α_1 α_2 and α_3 whose sum is zero.

$$\left.\begin{aligned}
M_{au} &= \Sigma\,(A_n \cos\alpha_1 \cos n\,\theta + A_n \sin\alpha_1 \sin n\,\theta)\\
M_{bv} &= \Sigma\,(B_n \cos\alpha_2 \cos n\,\theta + B_n \sin\alpha_2 \sin n\,\theta)\\
M_{cw} &= \Sigma\,(C_n \cos\alpha_3 \cos n\,\theta + C_n \sin\alpha_3 \sin n\,\theta)\\
M_{bw} &= \Sigma \left\{ C_n \cos\alpha_3 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + C_n \sin\alpha_3 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{cu} &= \Sigma \left\{ A_n \cos\alpha_1 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + A_n \sin\alpha_1 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{av} &= \Sigma \left\{ B_n \cos\alpha_2 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + B_n \sin\alpha_2 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{cv} &= \Sigma \left\{ B_n \cos\alpha_2 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + B_n \sin\alpha_2 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}\\
M_{aw} &= \Sigma \left\{ C_n \cos\alpha_3 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + C_n \sin\alpha_3 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}\\
M_{bu} &= \Sigma \left\{ A_n \cos\alpha_1 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + A_n \sin\alpha_1 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}
\end{aligned}\right\} \qquad (80)$$

Case V. Same as Case III except that the Primary Windings are Unsymmetrically disposed with respect to one another as well as being unequal.

$$
\left.
\begin{aligned}
M_{au} &= \Sigma\ (A_n \cos\alpha_1 \cos n\,\theta + A_n \sin\alpha_1 \sin n\,\theta) \\
M_{bv} &= \Sigma\ (B_n \cos\alpha_2 \cos n\,\theta + B_n \sin\alpha_2 \sin n\,\theta) \\
M_{cw} &= \Sigma\ (C_n \cos\alpha_3 \cos n\,\theta + C_n \sin\alpha_3 \sin n\,\theta) \\
M_{bw} &= \Sigma\ \left\{ B_n \cos\alpha_2 \cos n \left(\frac{2\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + B_n \sin\alpha_2 \sin n \left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{cu} &= \Sigma\ \left\{ A_n \cos\alpha_1 \cos n \left(\frac{2\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + A_n \sin\alpha_1 \sin n \left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{av} &= \Sigma\ \left\{ C_n \cos\alpha_3 \cos n \left(\frac{2\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + C_u \sin\alpha_3 \sin n \left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{cv} &= \Sigma\ \left\{ C_n \cos\alpha_3 \cos n \left(\frac{4\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + C_n \sin\alpha_3 \sin n \left(\frac{4\,\pi}{3} + \theta\right)\right\} \\
M_{aw} &= \Sigma\ \left\{ A_n \cos\alpha_1 \cos n \left(\frac{4\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + A_n \sin\alpha_1 \sin n \left(\frac{4\,\pi}{3} + \theta\right)\right\} \\
M_{bu} &= \Sigma\ \left\{ B_n \cos\alpha_2 \cos n \left(\frac{4\,\pi}{3} + \theta\right) \right. \\
&\qquad \left. + B_n \sin\alpha_2 \sin n \left(\frac{4\,\pi}{3} + \theta\right)\right\}
\end{aligned}
\right\} \tag{81}
$$

The expressions for dissymmetry in both windings and for unsymmetrically wound coils, etc., are more complicated and will be dealt with in the Appendix.

The impedances Z_{aa} Z_{bb}, etc., Z_{au} Z_{bv}, etc., are functions of M_{ac} M_{bb}, etc., M_{au} M_{bv}, etc., and the resistances of the system. The component of e. m. f. proportional to the current due to mutual impedance is so small that it may generally be neglected so that Z_{au} becomes $\frac{d}{dt} M_{au}$, $Z_{bv} = \frac{d}{dt} M_{bv}$ and so forth.

If the secondary winding is rotating at an angular velocity α, θ in equation (**55**) becomes $\alpha\, t$ and the operators Z_{aa}, etc. operate on such products as $e^{j\alpha t}\, \check{I}_{u1}$ $e^{j\alpha t}\, \check{I}_{u2}$ where $\check{I}_{u1}$ and $\check{I}_{u2}$ are three variables.

The following relations will be found useful in the application of the method in actual examples.

If D denotes the operator $\frac{d}{d\,x}$ and $\varphi\,(Z)$ is a rational algebraic function of Z

$$\left.\begin{aligned} \varphi\,(D)\, e^{ax} &= \varphi\,(a)\, e^{ax} \\ \varphi\,(D)\, \{e^{ax}\; X\} &= e^{ax}\; \varphi\,(D + a)\, X \\ \varphi\,(D)\; Y &= e^{ax}\, \varphi\; (D + a)\; Y\, e^{-ax} \end{aligned}\right\} \quad (\mathbf{82})$$

Where X and Y may be any function of x.

Star and Delta e.m.fs. and Currents in Terms of Symmetrical Components

It has been shown in the preceding portion of this paper that the e. m. fs. $\check{E}_a$ $\check{E}_b$ and $\check{E}_c$ and the currents $\check{I}_a$ $\check{I}_b$ and $\check{I}_c$ whatever their distortion, may be represented by the sum of symmetrical systems of e. m. fs. or currents so that the two expressions

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^0\, \check{E}_{a0} + S^1\, \check{E}_{a1} + S^2\, \check{E}_{a2} \\ S\,(\check{I}_a) &= S^0\, \check{I}_{a0} + S^1\, \check{I}_{a1} + S^2\, \check{I}_{a2} \end{aligned}\right\} \quad (\mathbf{83})$$

completely define these two systems.

If we take the delta e. m. fs. and currents corresponding to $S^0\, \check{E}_{a0}$, $S^1\, \check{E}_{a1}$ and $S^2\, \check{E}_{a2}$, $S^1\, \check{I}_{a1}$, $S^2\, \check{I}_{a2}$, we have, since $\check{E}_{bc1}$ leads $\check{E}_{a1}$ by $\frac{\pi}{2}$ and $\check{E}_{bc2}$ lags behind $\check{E}_{a2}$ by the same angle

$$\left.\begin{aligned} S^0\, \check{E}_{bc0} &= 0 \\ S^1\, \check{E}_{bc1} &= j\,\sqrt{3}\, S^1\, \check{E}_{a1} \\ S^2\, \check{E}_{bc2} &= -\,j\,\sqrt{3}\, S^2\, \check{E}_{a2} \\ S^0\, \check{I}_{bc0} &= \text{indeterminate from } S\,(\check{I}_a) \\ S^1\, \check{I}_{bc1} &= j\,\frac{1}{\sqrt{3}}\, S^1\, \check{I}_{a1} \\ S^2\, \check{I}_{bc2} &= -j\,\frac{1}{\sqrt{3}}\, S^2\, \check{I}_{a2} \end{aligned}\right\} \quad (\mathbf{84})$$

And therefore if we take $\check{E}_{ab}$ as the principal vector

$$\left.\begin{aligned} S^0\, \check{E}_{ab0} &= 0 \\ S^1\, E_{ab1} &= j\, a\, \sqrt{3}\, \check{E}_{a1} \\ S^2\, \check{E}_{ab2} &= -j\, a^2\, \sqrt{3}\, \check{E}_{a2} \\ S\, (\check{E}_{ab}) &= S^1\, \check{E}_{ab1} + S^2\, \check{E}_{ab2} \end{aligned}\right\} \quad (\mathbf{85})$$

The last equation of group (**85**) when expanded gives

$$\left.\begin{aligned} \check{E}_{ab} &= j\, \sqrt{3}\, (a\, \check{E}_{a1} - a^2\, \check{E}_{a2}) \\ \check{E}_{bc} &= j\, \sqrt{3}\, (\check{E}_{a1} - \check{E}_{a2}) \\ \check{E}_{ca} &= j\, \sqrt{3}\, (a^2\, \check{E}_{a1} - a\, \check{E}_{a2}) \end{aligned}\right\} \quad (\mathbf{86})$$

which may also be obtained direct from (**83**) by means of the relations

$$\begin{aligned} \check{E}_{ab} &= \check{E}_b - \check{E}_a \\ \check{E}_{bc} &= \check{E}_c - \check{E}_b \\ \check{E}_{ca} &= \check{E}_a - \check{E}_0 \end{aligned}$$

Similarly

$$\begin{aligned} S^0\, \check{I}_{ab} &= \text{indeterminate from } S\, (\check{I}_a) \\ S^1\, \check{I}_{ab1} &= j\, a\, \frac{1}{\sqrt{3}}\, \check{I}_{a1} \\ S^2\, \check{I}_{ab2} &= j\, a^2\, \frac{1}{\sqrt{3}}\, \check{I}_{a2} \\ S\, (\check{I}_{ab}) &= S^0\, \check{I}_{ab0} + S^1\, I_{ab1} + S^2\, \check{I}_{ab2} \end{aligned} \quad (\mathbf{87})$$

with similar expression for $\check{I}_{ab}$ $\check{I}_{bc}$ and $\check{I}_{ca}$ which may be verified by means of the relations

$$\begin{aligned} \check{I}_a &= \check{I}_{ca} - \check{I}_{ab} + \check{I}_{a0} \\ \check{I}_b &= \check{I}_{ab} - \check{I}_{bc} + \check{I}_{a0} \\ \check{I}_c &= \check{I}_{bc} - \check{I}_{ca} + \check{I}_{a0} \end{aligned}$$

Conversely to (**84**) we have the following relations

$$\left.\begin{aligned}
S^0\,\check{E}_{a0} &= \text{indeterminate from } S\,(\check{E}_{ab})\\
S^1\,\check{E}_{a1} &= -\,j\frac{1}{\sqrt{3}}\,S^1\,\check{E}_{bc1} = -j\,\frac{a^2}{\sqrt{3}}\,S^1\,\check{E}_{ab1}\\
S^2\,\check{E}_{a2} &= \;\;j\,\frac{1}{\sqrt{3}}\,S^2\,E_{bc2} = \;\;j\,\frac{a}{\sqrt{3}}\,S^2\,E_{ab2}\\
S^0\,\check{I}_{a0} &= \text{indeterminate from } S\,(\check{I}_{ab})\\
S^1\,\check{I}_{a1} &= -\,j\,\sqrt{3}\,S^1\,I_{bc1} = -\,j\,a^2\,\sqrt{3}\,S^1\,\check{I}_{ab1}\\
S^2\,\check{I}_{a2} &= \;\;j\,\sqrt{3}\,S^2\,\check{I}_{bc} = \;\;j\,a\,\sqrt{3}\,S^2\,\check{I}_{ab2}
\end{aligned}\right\}\qquad(\mathbf{88})$$

It will be sufficient in order to illustrate the application of the principle of symmetrical coordinates to simple circuits to apply it to a few simple cases of transformer connections before proceeding to its application to rotating polyphase systems to which it is particularly adapted.

Unsymmetrical Bank of Delta-Delta Transformers Operating on a Symmetrical Circuit Supplying a Balanced System

Let the transformer effective impedances be $Z_{AB}\ Z_{BC}\ Z_{CA}$ and let the secondary load currents be $\check{I}_U\ \check{I}_V$ and $\check{I}_W$ and let the star load impedance be Z. One to one ratio of transformation will be assumed, and the effect of the magnetizing current will be neglected. The symmetrical equations are

$$\left.\begin{aligned}
O &= S^0\,(Z_{AB0}\,\check{I}_{ab0} + Z_{AB2}\,\check{I}_{ab1} + Z_{AB1}\,\check{I}_{ab2})\\
S^1\,\check{E}_{uv1} &= S^1\,\check{E}_{ab1} - S^1\,(Z_{AB1}\,\check{I}_{ab0} + Z_{AB0}\,\check{I}_{ab1} + Z_{AB2}\,\check{I}_{ab2})\\
S^2\,\check{E}_{uv2} &= 0 - S^2\,(Z_{AB2}\,\check{I}_{ab0} + Z_{AB1}\,\check{I}_{ab1} + Z_{AB0}\,\check{I}_{ab2})\\
S^0\,\check{I}_{u0} &= 0\\
S^1\,Z\,\check{I}_{u1} &= \check{E}_{u1}\\
S^2\,Z\,\check{I}_{u2} &= \check{E}_{u2}
\end{aligned}\right\}\qquad(\mathbf{89})$$

Since the transformation ratio is unity and the effects of magnetizing currents are negligible $S^1\,\check{I}_{ab1} = S^1\,\check{I}_{UV1}$, $S^2\,\check{I}_{ab2} = S^2\,\check{I}_{UV2}$. And therefore by means of the relations (**85**), the last two equations may be expressed

$$\left.\begin{aligned} S^1\,\check{E}_{uv1} &= S^1\,3\,Z\,\check{I}_{ab1} \\ S^2\,\check{E}_{uv2} &= S^2\,3\,Z\,\check{I}_{ab2} \end{aligned}\right\} \quad (\mathbf{90})$$

in other words, the symmetrical components appear in the secondary as independent systems, $3\,Z$ being the delta load impedance equivalent to the star impedance Z.

Substituting from (**90**) in the second and third equation and eliminating $\check{I}_{ab0}$ by means of the first equation, and we have

$$\left.\begin{aligned} S^1\,\check{E}_{ab1} &= S^1\left\{\left(3\,Z + Z_{AB0} - \frac{Z_{AB1}\,Z_{AB2}}{Z_{AB0}}\right)\check{I}_{ab1}\right. \\ &\qquad \left. + \left(Z_{AB2} - \frac{Z_{AB1}^2}{Z_{AB0}}\right)\check{I}_{ab2}\right\} \\ S^2\,O &= S^2\left\{\left(Z_{AB1} - \frac{Z^2_{AB2}}{Z_{AB0}}\right)\check{I}_{ab1}\right. \\ &\qquad \left. + \left(3\,Z + Z_{AB0} - \frac{Z_{AB1}\,Z_{AB2}}{Z_{AB0}}\right)\check{I}_{ab2}\right\} \end{aligned}\right\} \quad (\mathbf{91})$$

which, when S^1 and S^2 are removed, give two simultaneous equations in $\check{I}_{ab1}$ and $\check{I}_{ab2}$.

A modification of the problem may occur even when the load impedances are symmetrical, as they may have symmetrical but unequal impedances Z_1 and Z_2, to the two components $\check{I}_{v1}$ and $\check{I}_{v2}$ respectively, as in the case of a load consisting of a symmetrical rotating machine. The equations corresponding to (**89**), (**90**) and (**91**) then become

$$\left.\begin{aligned} O &= S^0\,(Z_{AB0}\,\check{I}_{ab0} + Z_{AB2}\,\check{I}_{ab1} + Z_{AB1}\,\check{I}_{ab2}) \\ S^1\,\check{E}_{uv1} &= S^1\,\check{E}_{ab1} - S^1\,(Z_{AB1}\,\check{I}_{ab0} + Z_{AB0}\,\check{I}_{ab1} + Z_{AB2}\,\check{I}_{ab2}) \\ S^2\,E_{uv2} &= O - S^2\,(Z_{AB2}\,\check{I}_{ab0} + Z_{AB1}\,\check{I}_{ab1} + Z_{AB0}\,\check{I}_{ab2}) \\ S^0\,\check{I}_{u0} &= O \\ S^1\,Z_1\,\check{I}_{u1} &= \check{E}_{u1} \\ S^2\,Z_2\,\check{I}_{u2} &= \check{E}_{u2} \end{aligned}\right\} \quad (\mathbf{92})$$

$$\left.\begin{aligned} S^1\,\check{E}_{uv1} &= S^1\,3\,Z_1\,\check{I}_{ab1} \\ S^2\,\check{E}_{uv2} &= S^2\,3\,Z_2\,\check{I}_{ab2} \end{aligned}\right\} \quad (\mathbf{93})$$

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S^1 \left\{ \left(3 Z_1 + Z_{AB0} - \frac{Z_{AB1} Z_{AB2}}{Z_{AB0}} \right) \check{I}_{ab1} \right. \\ &\quad \left. + \left(Z_{AB2} - \frac{Z^2_{AB1}}{Z_{AB0}} \right) \check{I}_{ab2} \right\} \\ S^2 O &= S^2 \left\{ \left(Z_{AB1} - \frac{Z^2_{AB1}}{Z_{AB0}} \right) \check{I}_{ab1} \right. \\ &\quad \left. + \left(3 Z_2 + Z_{AB0} - \frac{Z_{AB1} Z_{AB2}}{Z_{AB0}} \right) \check{I}_{ab2} \right\} \end{aligned} \right\} \quad (94)$$

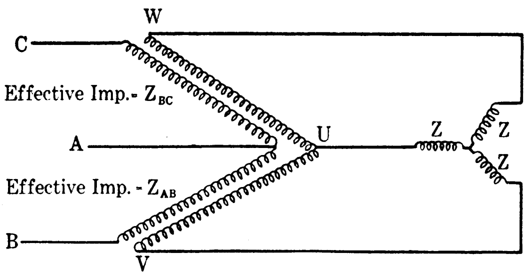

FIG. 4—OPEN DELTA OR V CONNECTION.

In an open delta system $Z_{AB1} = Z_{AB2} = Z_{AB0} - Z_{AB}$ the transformers in this case being both the same, equation (**91**) becomes in this particular case where Z_{AB0} is infinite

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S^1 \{ (3 Z + 2 Z_{AB}) \check{I}_{ab} + Z_{AB} \check{I}_{ab2} \} \\ S^2 O &= S^2 \{ Z_{AB} \check{I}_{ab1} + (3 Z + 2 Z_{AB}) \check{I}_{ab2} \} \end{aligned} \right\} \quad (95)$$

and we have

$$\check{I}_{ab0} = - \check{I}_{ab1} - I_{ab2} \quad (96)$$

Similarly, instead of (**94**) we have

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S \{ (3 Z_1 + 2 Z_{AB}) \check{I}_{ab1} + Z_{AB} \check{I}_{ab2} \} \\ S^2 O &= S^2 \{ Z_{AB} \check{I}_{ab1} + (3 Z_2 + 2 Z_{AB}) \check{I}_{ab2} \} \end{aligned} \right\} \quad (97)$$

The secondary voltages are obtained from (**90**) and (**93**) for this latter case.

The solution of (**95**) gives

$$\left.\begin{aligned}
\check{I}_{ab1} &= \frac{3\,Z_1 + 2\,Z_{AB}}{(3\,Z_1 + 3\,Z_{AB})\;(3\,Z_1 + Z_{AB})}\;\check{E}_{ab}\\
\check{I}_{ab2} &= -\;\frac{Z_{AB}}{(3\,Z_1 + 3\,Z_{AB})\;(3\,Z_1 + Z_{AB})}\;\check{E}_{ab}\\
\check{I}_{ab0} &= -\;\frac{1}{3\,Z_1 + 3\,Z_{AB}}\;\check{E}_{ab}
\end{aligned}\right\}\quad(\mathbf{98})$$

And we have

$$\left.\begin{aligned}
S^1\,\check{I}_{a1} &= S^1\;\frac{3\,Z_1 + 2\,Z_{AB}}{3\,(Z_1 + Z_{AB})\left(Z_1 + \dfrac{Z_{AB}}{3}\right)}\;\check{E}_a\\
S^2\,\check{I}_{a2} &= S^2\;\frac{Z_{AB}}{3\,(Z_1 + Z_{AB})\left(Z_1 + \dfrac{Z_{AB}}{3}\right)}\;\check{E}_b
\end{aligned}\right\}\quad(\mathbf{99})$$

And therefore

$$\left.\begin{aligned}
\check{I}_a &= \frac{\check{E}_a}{Z_1 + \dfrac{Z_{AB}}{3}} + \frac{\frac{1}{3}\,Z_{ab}}{(Z_1 + Z_{AB})\left(Z_1 + \dfrac{Z_{AB}}{3}\right)}\check{E}_{ab}\\
\check{I}_b &= \frac{\check{E}_b}{Z_1 + \dfrac{Z_{AB}}{3}} - \frac{\frac{1}{3}\,Z_{AB}}{(Z_1 + Z_{AB})\left(Z_1 + \dfrac{Z_{AB}}{3}\right)}\check{E}_{ab}\\
\check{I}_c &= \frac{\check{E}_c}{Z_1 + \dfrac{Z_{AB}}{3}}
\end{aligned}\right\}\quad(\mathbf{100})$$

Three Phase System with Symmetrical Waves Having Harmonics

We may express $\check{E}_a$ in the following form:

$$\left.\begin{aligned}
\check{E}_a &= E_1\,e^{jwt} + E_2\,e^{j2wt} + E_3\,e^{j3wt} + \;.\;.\\
&= \Sigma\,E_n\,e^{jnwt}
\end{aligned}\right\}\quad(\mathbf{101})$$

where E_n is in general a complex number.

If the system is symmetrical three-phase $\check{E}_b$ is obtained by displacing the complete wave by the angle $-\dfrac{2\,\pi}{3}$ or

$$\check{E}_b = e^{-j\frac{2\pi}{3}} E_1 e^{jw\tau} + e^{-j\frac{4\pi}{3}} E_2 e^{j2w\tau} + e^{-j\frac{6\pi}{3}} E_3 e^{j3w\tau} + \ldots$$

$$E_c = e^{j\frac{2\pi}{3}} E_1 e^{jw\tau} + e^{j\frac{4\pi}{3}} E_2 e^{j2w\tau} e^{j\frac{6\pi}{3}} E_3 e^{j3w\tau} + \ldots$$

or since $e^{-j\frac{2\pi}{3}} = a^2$, $e^{j\frac{2\pi}{3}} = a$ etc.

$$\left.\begin{aligned} \check{E}_a &= E_1 e^{jwt} + E_2 e^{j2wt} + E_3 e^{j3wt} + \ldots \\ \check{E}_b &= a^2 E_1 e^{jwt} + a E_2 e^{j2wt} + E_3 e^{j3wt} + \ldots \\ \check{E}_c &= a E_1 e^{jwt} + a^2 E_2 e^{j\tau wt} + E_3 e^{j3wt} + \ldots \end{aligned}\right\} \quad (102)$$

or

$$\left.\begin{aligned} S(\check{E}_a) = S^0 &\{E_3 e^{j3wt} + E_6 e^{j6wt} + E_9 e^{j9wt} + \ldots\} \\ + S^1 &\{E_1 e^{jwt} + E_4 e^{j4wt} + E_7 e^{j7wt} + \ldots\} \\ + S^2 &\{E_2 e^{j2wt} + E_5 e^{j5wt} + E_8 e^{j8wt} + \ldots\} \end{aligned}\right\} \quad (103)$$

$$S(\check{E}_a) = S^0 \Sigma (E_{3n} e^{j3nwt}) + S^1 \Sigma (E_{3n-2} e^{j(3n-2)w\tau}) + S^2 \Sigma (E_{3n-1} e^{j(3n-1)w\tau}) \quad (104)$$

This shows that a symmetrical three-phase system having harmonics is made up of positive and negative phase sequence harmonic systems and others of zero phase sequence, that is to say of the same phase in all windings, which comprise the group of third harmonics. These facts are not generally appreciated though they are factors that may have an appreciable influence in the performance of commercial machines. It should be particularly noted that in three phase generators provided with dampers the fifth, eleventh, seventeenth, and twenty-third harmonics produce currents in the damper windings.

In dealing with the complex variable it will be convenient to use for the amplitude the root mean square value for each harmonic. When instantaneous values are required, the real part of the complex variable should be multiplied by $\sqrt{2}$. In the remainder of this paper this convention will be adopted.

Power Presentation in Symmetrical Co-ordinates

Since the power in an alternating current system is also a harmonically varying scalar quantity, it may therefore be represented in the same manner as the current or electromotive force, that is to say by a complex variable which we shall denote by $(P + j\,Q) + P_H + j\,Q_H)$ $P + j\,Q$ being the mean value, is the term of the complex variable of zero frequency, P representing the real power and Q the wattless power $\sqrt{P^2 + Q^2}$ will be the volt-amperes.

The value of the complex variable $(P + j\,Q) + (P_H + j\,Q_H)$ may be taken as

$$(P + j\,Q) + (P_H + j\,Q_H) = \check{E}\,\hat{I} + \check{E}\,\check{I} \tag{105}$$

with the provision that for all terms having negative indices the conjugate terms must be substituted, these terms being present in the product $\hat{E}\,\check{I} + \hat{E}\,\hat{I}$, which is the conjugate of the product (**105**). A similar rule holds good for the symmetrical vector system

$$\left.\begin{aligned} S(\check{E}_a) &= S^0\,\check{E}_{a0} + S^1\,\check{E}_{a1} + \ \ldots\ S^{n-1}\,E_{a(n-1)} \\ S(\check{I}_a) &= S^0\,\check{I}_{a0} + S^1\,\check{I}_{a1} + \ \ldots\ S^{n-1}\,I_{a(n-1)} \end{aligned}\right\} \tag{106}$$

The conjugate of $S\,\check{I}_a$ is

$$S(\hat{I}_a) = S^0\,\hat{I}_{a0} + S^{(n-1)}\,\hat{I}_{a1} + \ \ldots\ S^1\,\hat{I}_{a(n-1)} \tag{107}$$

and the Power is represented by

$$(P + P_b) + j\,(Q + Q_b) = \Sigma\,\{S(\check{E}_a)\,S(\hat{I}_a) + S(\check{E}_a)\,S(\check{I}_a)\} \tag{108}$$

with the same provision for terms having negative indices the sign Σ signifies that all the products in each sequence are added together.

$$\left.\begin{aligned} \Sigma\,\{S(\hat{I}_a)\,S(\check{E}_a)\} = {} & \Sigma\,S^0\,\{\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \ \ldots \\ & \qquad \hat{I}_{a(n-1)}\,\check{E}_{a(n-1)}\} \\ & + \Sigma\,S^1\,\{\hat{I}_{a0}\,\check{E}_{a1} + \hat{I}_{a1}\,\check{E}_{a2} + \hat{I}_{a2}\,\check{E}_{a3} + \ \ldots \\ & \qquad \hat{I}_{a(n-1)}\,\check{E}_{a0}\} \\ & + \Sigma\,S^2\,\{\hat{I}_{a0}\,\check{E}_{a2} + \hat{I}_{a1}\,\check{E}_{a3} + \hat{I}_{a2}\,\check{E}_{4} + \ \ldots \\ & \qquad \check{I}_{a(n-1)}\,\check{E}_{a1}\} \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \Sigma\,S^{(n-1)}\,\{\hat{I}_{a0}\,\check{E}_{a(n-1)} + \hat{I}_{a1}\,\check{E}_{a0} + \ \ldots \\ & \qquad + \hat{I}_{a(n-1)}\,\check{E}_{a(n-2)}\} \end{aligned}\right\} \tag{109}$$

The terms prefixed by S^1, S^2, S^3 . . . $S^{(n-1)}$ all become zero and since S^0 becomes n

$$\Sigma\, S\,(\hat{I}_a)\; S\,(\check{E}_a) \;=\; n\;\{\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \quad . \quad . \quad . \quad \hat{I}_{a(n-1)}\;\check{E}_{a(n-1)}\} \tag{110}$$

In a similar manner it may be shown that

$$\Sigma\, S\,(\check{I}_a)\; S\,(\check{E}_a) \;=\; n\;\{\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a(n-1)} + \check{I}_{a2}\,\check{E}_{a(n-2)} + \quad . \quad . \quad . \quad \check{I}_{a(n-1)}\;\;\check{E}_{a1}\} \tag{111}$$

and therefore

$$\begin{aligned}(P + j\,Q) + (P_{\text{H}} + j\,Q_{\text{H}}\,) \;=\; & n\;\{\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \quad . \quad . \quad . \quad \hat{I}_{a(n-1)}\;\check{E}_{a(n-1)}\} \\ & + n\;\{\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a(n-1)} + \quad . \quad . \quad . \quad \check{I}_{a(n-1)}\check{E}_{a1}\}\end{aligned} \tag{112}$$

For a three-phase system the expression reduces to

$$\begin{aligned}(P + j\,Q) + (P_{\text{H}} + j\,Q_b) \;=\; & 3\;(\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \hat{I}_{a2}\,\check{E}_{a2}) \\ & + 3\;(\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a2} + \check{I}_{a2}\,\check{E}_{a1})\end{aligned} \tag{113}$$

In the above expression $P + P_{\text{H}}$ is the value of the instantaneous power on the system, P being the mean value and P_{H} the harmonic portion. When the currents are simple sine waves, Q may be interpreted to be the mean wattless power of the circuit or the sum of the wattless voltamperes of each circuit. In rotating machinery since the coefficients of mutual induction may be complex harmonic functions of the angular velocity, this is not strictly true for all cases; but if the effective impedances to the various frequencies of the component currents be used, it will be found to be equal to the mean wattless voltamperes of the system with each harmonic considered independent.

In a balanced polyphase system P_{H} and Q_{H} both become zero.

The instantaneous power is a quantity of great importance in polyphase systems because the instantaneous torque is proportional to it and this quantity enters into the problem of vibrations which is at times a matter of great importance, especially when caused by unbalanced e. m. fs. A system of currents and e. m. fs. may be transformed to balanced polyphase by means of transformers alone, provided that the value of P_{H} is zero, while on the other hand polyphase power cannot be supplied from a pulsating power system without means for supplying the necessary storage to make a continuous flow of energy.

PART II

Application of the Method to Rotating Polyphase Networks

The methods of determining the constants Z_a Z_u, M, etc., of co-axial cylindrical networks is taken up in Appendix I of this paper. It will be assumed that the reader has familiarized himself with these quantities and understands their significance. We shall first consider the case of symmetrically wound machines taking up the simple cases first and proceeding to more complex ones.

Symmetrically Wound Induction Motor Operating on Unsymmetrical Polyphase Circuit

Denoting the pole pitch angle by π let the synchronous angular velocity be ω_0 and let the angular slip velocity be ω_1. And let $S^1\,E_{a1}$ $S^2\,E_{a2}$ be the symmetrical components of impressed polyphase e. m. f. Let R_a be the primary resistance and R_u the secondary resistance. The primary self-inductance being M_{aa}, that of the secondary being M_{uu} and corresponding symbols being used to denote the mutual inductances between the different pairs of windings. Then by means of (**39**), (**40**), (**56**) and (**57**)

$$\left.\begin{aligned}
S^1\,\check{E}_a{}^1 &= S^1\left\{R_a\,\check{I}_{a1} + 1\tfrac{1}{2}\,M_{aa}\,\frac{d}{d\,t}\,\check{I}_{a1}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\,e^{j\,(w_0 - w_1)t}\,\check{I}_{u1}\right\}\\
S^2\,\check{E}_{a2} &= S^2\left\{R_a\,\check{I}_{a2} + 1\tfrac{1}{2}\,M_{aa}\,\frac{d}{d\,t}\,\check{I}_{a2}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\,e^{-j(w_0 - w_1)t}\,\check{I}_{u2}\right\}\\
S^1\,\check{E}_{u1} = O &= S^1\left\{R_u\,\check{I}_{u1} + 1\tfrac{1}{2}\,M_{uu}\,\frac{d}{d\,t}\,\check{I}_{u1}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\,e^{-j(w_0 - w_1)t}\,\check{I}_{a1}\right\}\\
S^2\,\check{E}_{u2} = O &= S^2\left\{R_u\,\check{I}_{u2} + 1\tfrac{1}{2}\,M_{uu}\,\frac{d}{d\,t}\,\check{I}_{u2}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\,e^{j\,(w_0 - w_1)t}\,\check{I}_{a2}\right\}
\end{aligned}\right\} \quad (\mathbf{114})$$

denote $1\frac{1}{2}\ M_{aa}$ by L_a and $1\frac{1}{2}\ M_{uu}$ by L_u, $1\frac{1}{2}\ M_{au}$ by M, the equations (**1**) become

$$\left.\begin{aligned}
S^1\,\check{E}_{a1} &= S^1 \left\{ \left(R_a + L_a \frac{d}{d\,t}\right) \check{I}_{a1} \right. \\
&\qquad \left. + M \frac{d}{d\,t}\, e^{j\,(w_0 - w_1)t}\, \check{I}_{u1} \right\} \\
S^2\,\check{E}_{a2} &= S^2 \left\{ \left(R_a + L_a \frac{d}{d\,t}\right) \check{I}_{a2} \right. \\
&\qquad \left. + M \frac{d}{d\,t}\, e^{-j(w_0 - w_1)t}\, \check{I}_{u2} \right\} \\
S^1\,\check{E}_{u1} = O &= S^1 \left\{ \left(R_u + L_u \frac{d}{d\,t}\right) \check{I}_{u1} \right. \\
&\qquad \left. + M \frac{d}{d\,t}\, e^{-j\,(w\ - w_1)t}\, \check{I}_{a1} \right\} \\
S^2\,\check{E}_{u2} = O &= S^2 \left\{ \left(R_u + L_u \frac{d}{d\,t}\right) \check{I}_{u2} \right. \\
&\qquad \left. + M \frac{d}{d\,t}\, e^{j\,(w_0 - w_1)t}\, \check{I}_{a2} \right\}
\end{aligned}\right\} \qquad (\mathbf{115})$$

From the last two equations we have

$$\check{I}_{u1} = - \frac{M \dfrac{d}{d\,t}}{R_u + L_u \dfrac{d}{d\,t}}\, e^{-j\,(w_0 - w_1)\cdot}\, \check{I}_{a1} \qquad (\mathbf{116})$$

$$\check{I}_{u2} = - \frac{M \dfrac{d}{d\,t}}{R_u + L_u \dfrac{d}{d\,t}}\, e^{j\,(w_0 - w_1)t}\, \check{I}_{a2} \qquad (\mathbf{117})$$

Substituting these in the first two equations of (**115**) we obtain

$$S'\,\check{E}_{a1} = S^1 \left[\left(R_a + L_a \frac{d}{d\,t}\right) - \frac{M^2 \dfrac{d}{d\,t} \left\{ \dfrac{d}{d\,t} - j\,(w_0 - w_1) \right\}}{R_u + L_u \left\{ \dfrac{d}{d\,t} - j\,(w_0 - w_1) \right\}} \right] \check{I}_{a1} \qquad (\mathbf{118})$$

$$S^2 \check{E}_{a2} = S^2 \left[\left(R_a + L_a \frac{d}{d\,t} \right) - \frac{M^2 \frac{d}{d\,t} \left\{ \frac{d}{d\,t} + (j\,w_0 - w_1) \right\}}{R_u + L_u \left\{ \frac{d}{d\,t} - j\,(w_0 - w_1) \right\}} \right] \check{I}_{a2} \quad (119)$$

If $\check{E}_{a1} = E_{a1}\, e^{jwt}$ and $\check{E}_{a2} = E_{a2}\, e^{jwt}$ the solution for $\check{I}_{a1}$ and $\check{I}_{a2}$ will be

$$\check{I}_{a1} = \frac{\check{E}_{a1}}{Z_1} \quad (120)$$

$$\check{I}_{a2} = \frac{\check{E}_{a2}}{Z_2} \quad (121)$$

Where

$$Z_1 = R_a + j\,w_0\,L_a + \frac{w_0\,w_1\,M^2}{R_u^2 + w_1^2\,L_u^2}\,(R_u - j\,w_1\,L_u) \quad (122)$$

$$Z_2 = R_a + j\,w_0\,L_a + \frac{w_0\,(2\,w_0 - w_1)\,M^2}{R_u^2 + (2\,w_0 - w_1)^2\,L_u^2}\,\{R_u - j\,(2\,w_0 - w_2)\,L_u\} \quad (123)$$

The impedances Z_1 and Z_2 will be found more convenient to use in the form

$$Z_1 = (R_a + K_1^2\,R_u) + j\,w_0\,(L_a - K_1^2\,L_u) + \frac{w_0 - w_1}{w_1}\,K_1^2\,R_u \quad (124)$$

$$Z_2 = (R_a + K_2^2\,R_u) + j\,w_0\,(L_a - K_2^2\,L_u) - \frac{w_0 - w_1}{Z\,w_0 - w_1}\,K_2^2\,R_u \quad (125)$$

Where, as we will see later, K_1^2 and K_2^2 are the squares of the transformation ratios between primary and secondary currents of positive and negative phase sequence.

The last real term in each expression is the virtual resistance due to mechanical rotation and when combined with the mean square current represents mechanical work performed, the positive sign representing work performed and the negative sign work required.

Thus, for example, to enable the currents $S^2\,\check{I}_{a2}$ to flow, the mechanical work $3\,I_{a2}^2\,\frac{w_0 - w_1}{2\,w_0 - w_1}\,K_1^2\,R_u$ must be applied to the shaft of the motor.

The phase angles of the symmetrical systems $S^1 \check{I}_{a1}$ $S^2 \check{I}_{a2}$ with respect to their impressed e. m. f., $S^1 \check{E}_{a1}$ and $S^2 \check{E}_{a2}$ are given by these impedances so that the complete solution of the primary circuit is thus obtained.

The secondary currents are given by equations (**116**) and (**117**) and are

$$\check{I}_{u1} = - \frac{j\, w_1\, M}{R_u + j\, w_1\, L_u}\, I_{a1}\, e^{j w_1 T} = \check{K}_1\, I_{a1} e^{j w_1 T} \tag{126}$$

$$\check{I}_{u2} = - \frac{j\,(2\, w_0 - w_1)\, M}{R_u + j\,(2\, w_0 - w_1)\, L_u}\, I_{a2}\, e^{j(2w_0 - w_1)T} = \check{K}_2\, I_{a2}\, e^{j(2w_0 - w_1)T} \tag{127}$$

In the results just given, M is not the maximum value of mutual inductance between a pair of primary and secondary windings but is equal to the total mutual inductance due to a current passing through the two coils W and V through the coil

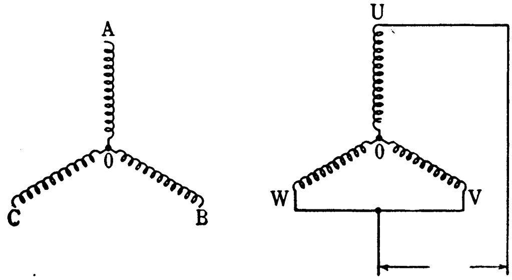

FIG. 5

U as shown in the sketch Fig 5 and the winding "A" when A and U have their planes of symmetry coincident.

Where the windings are symmetrical the induced e. m. f. is independent of the division of current between W and V, but this quantity must not be used in unsymmetrical windings, or with star windings having a neutral point connection so that $\check{I}_{a0}$ is not zero.

The appearance of M in this equation follows from the equation

$$\check{I}_u + \check{I}_v + \check{I}_w = 0$$

so that

$$\check{I}_u = - (\check{I}_v + \check{I}_w)$$

The power delivered by the motor is

$$P_0 = 3 \left\{ \frac{w_0 - w_1}{w_1} K_1^2\, I_{a1}^2\, R_u - \frac{w_0 - w_1}{2\, w_0 - w_1} K_2^2\, I_{a2}^2\, R_u \right\} \tag{128}$$

The copper losses are given by

$$P_L = 3\,\{I_{a1}^2\,(R_P + K_1^2\,R_u) + I_{a2}^2\,(R_P + K_2^2\,R_u)\} \qquad (129)$$

The iron loss is independent of the copper loss and power output. The iron loss and windage may be taken as

$$P_F = \text{Iron loss and windage} \qquad (130)$$

The power input as

$$P_I = P_O + P_L + P_F \qquad (131)$$

The mechanical power output is P_O less friction and windage losses.

$$\text{Torque} = 3\left\{\frac{1}{w_1}\,K_1^2\,I_{a1}^2\,R_u - \frac{1}{2\,w_0 - w_1}\,K_2^2\,I_{a2}^2\,R_u\right\} \times 10^7 \text{ dyne-cm.} \qquad (132)$$

The kv-a. at the terminals is

$$\sqrt{P_I^2 + Q_I^2} = \text{The effective value of } 3\,(E_{a1}\,I_{a1} + E_{a2}\,I_{a2}) \qquad (133)$$

This last result may be arrived at in the following way

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2}) \\ S\,(\hat{I}_a) &= S^2\,\hat{I}_{a1} + S^1\,\hat{I}_{a2} \end{aligned}\right\} \qquad (134)$$

Since $S^2\hat{I}_{a1}$ is conjugate to $S^1\,\check{I}_{a1}$, etc.

The product of $\check{E}_{sa}$ and $\hat{I}_{sa}$ is the power product of the two vectors, $S\,(\check{E}_a)$ and $S\,(\check{I}_a)$ and omits the harmonic variation as a double frequency quantity, the average wattless appears as an imaginary non-harmonic quantity.

$$P_I + j\,Q_I\,\Sigma\,(S^0\,\check{E}_{a1}\,\hat{I}_{a1} + S^0\,\check{E}_{a2}\,\hat{I}_{a2} + S^1\,\check{E}_{a2}\,\hat{I}_{a1} + S^2\,\check{E}_{a1}\,\hat{I}_{a2}) \qquad (135)$$

The S^1 and S^2 products have zero values, since the sum of the terms of each sequence is zero, hence—

$$P_I + j\,Q_I = 3\,(\check{E}_{a1}\,\hat{I}_{a1} + \check{E}_{a2}\,\hat{I}_{a2}) \qquad (136)$$

$$\sqrt{P_I^2 + Q_I^2} = \text{The effective value of } 3\,(\check{E}_{a1}\,\hat{I}_{a1} + \check{E}_{a2}\,\hat{I}_{a2}) \qquad (137)$$

The solution for the general case of symmetrical motor operating on an unsymmetrical circuit is not of as much interest as certain special cases depending thereon. Some of the most important of these will be taken up in the following paragraphs.

Case I. Single-Phase e. m. f. Impressed across one phase of three-phase motor.

Assuming the single-phase voltage to be $\check{E}_{bc}$ impressed across the terminals B C. The known data or constraints are

$$\left.\begin{aligned} \check{E}_{bc} &= j \sqrt{3}\,(\check{E}_{a1} - \check{E}_{a2}) \\ \check{I}_a &= O,\ \check{I}_b = -\check{I}_c \end{aligned}\right\} \qquad (138)$$

and therefore

$$\check{I}_{a1} = -\check{I}_{a2} \qquad (139)$$

$$\frac{\check{E}_{a1}}{Z_1} = -\frac{\check{E}_{a2}}{Z_2}$$

$$\check{E}_{a2} = -\frac{Z_2}{Z_1}\check{E}_{a1} \qquad (140)$$

Substituting in (**138**)

$$\left.\begin{aligned} \check{E}_{a1} &= -j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{Z_1}{Z_1 + Z_2} \\ \check{E}_{a2} &= j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{Z_2}{Z_1 + Z_2} \end{aligned}\right\} \qquad (141)$$

and therefore

$$\left.\begin{aligned} \check{I}_{a1} &= -j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{1}{Z_1 + Z_2} \\ \check{I}_{a2} &= j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{1}{Z_1 + Z_2} \end{aligned}\right\} \qquad (142)$$

Since $\check{I}_b = \check{I}_{b1} + \check{I}_{b2} = a^2\,\check{I}_{a1} + a\,\check{I}_{a2}$

$$\check{I}_b = -I_c = -\frac{\check{E}_{bc}}{Z_1 + Z_2} \qquad (143)$$

$$P_0 = \left(\frac{w_0 - w_1}{w_1} K_2^2 R_u - \frac{w_0 - w_1}{2\,w_0 - w_1} K_2^2 R_u\right) I_0^2 \qquad (144)$$

$$P_1 + j\,Q_1 = I_b^2\,(Z_1 + Z_2) + P_F \qquad (145)$$

The power factor is obtained from (**145**) by the formula

$$\cos\alpha = \frac{P_1}{\sqrt{P_1^2 + Q_1^2}} \qquad (146)$$

Substituting from (**142**) in equation (**126**) and (**127**) of the general case we obtain for the secondary currents

$$\left.\begin{aligned} \check{I}_{u1} &= - j\,\check{K}_1 \frac{E_{bc}}{Z_1 + Z_2}\, e^{j w_1 t} \\ \check{I}_{u2} &= j\,\check{K}_2 \frac{E_{bc}}{Z_1 + Z_2}\, e^{j(2w_0 - w_1)t} \end{aligned}\right\} \quad (\mathbf{147})$$

Many unsymmetrical cases may be expressed in terms of the operation of coupled symmetrical motors operating on symmetrical systems. This is invariably the case with symmetrical polyphase motors operating on single phase circuits. Since the physical interpretations are useful in impressing the facts on ones memory they will be given whenever they appear to be useful.

Equations (**141**) and (**142**) show that single-phase operation is exactly equivalent to operating two duplicate motors in series with a symmetrical polyphase e. m. f. $S^1 E_{ab}$ impressed across one motor, the other being connected in series with the first but with phase sequence reversed, the two motors being directly coupled.

Case II. B and C connected together e. m. f. impressed across A B.

The data given by the conditions of constraint are

$$\left.\begin{aligned} \check{E}_{ab} &= - \check{E}_{ca} \\ \check{E}_{bc} &= 0 = j\sqrt{3}\,(\check{E}_{a1} - \check{E}_{a2}) \end{aligned}\right\} \quad (\mathbf{148})$$

We therefore have

$$\check{E}_{a1} = \check{E}_{a2} = - \frac{\check{E}_{ab}}{3} \quad (\mathbf{149})$$

and

$$\left.\begin{aligned} \check{I}_{a1} &= - \frac{\check{E}_{ab}}{3\,Z_1} \\ \check{I}_{a2} &= - \frac{\check{E}_{ab}}{3\,Z_2} \end{aligned}\right\} \quad (\mathbf{150})$$

The remainder follows from the general solution and need not be repeated here.

(**150**) shows that a motor operated in this manner is the exact equivalent in all respects to two duplicate mechanically coupled polyphase motors, one of which has sequence reversed, operating in parallel on a balanced three-phase circuit of e. m. f. $S^1 \frac{E_{ab}}{\sqrt{3}}$.

The secondary currents follow from substitution of (**150**) in equations (**126**) and (**127**) of the general case.

Case III. B and C connected together by the terminals of a balance coil, the impressed e. m. f. E_{AD} *applied between A and the middle point of the balance coil. Resistance and reactance of balance coil negligible.*

The data furnished by the connection in this case is

$$\check{I}_b = \check{I}_c = - \frac{\check{I}_a}{2} \tag{151}$$

and therefore

$$\check{I}_{a1} = \frac{\check{I}_a - a \dfrac{\check{I}_a}{2} - a^2 \dfrac{\check{I}_a}{2}}{3} = \frac{\check{I}_a}{2}$$

$$\check{I}_{a2} = \check{I}_{a1} = \frac{\check{I}_a}{2}$$

We therefore have

$$\left.\begin{aligned} \check{E}_{a1} &= \frac{Z_1 \check{I}_a}{2} \\ \check{E}_{a2} &= \frac{Z_2 \check{I}_a}{2} \end{aligned}\right\} \tag{152}$$

we have

$$\check{E}_{ab} = j \sqrt{3} \, (a \, \check{E}_{a1} - a^2 \, \check{E}_{a2})$$

$$= j \sqrt{3} \, \frac{\check{I}_a}{2} \, (a \, Z_1 - a^2 \, Z_2)$$

$$\check{E}_{bc} = j \sqrt{3} \, \frac{\check{I}_a}{2} \, (Z_1 - Z_2)$$

$$\left.\begin{aligned} \check{E}_{ad} &= \left(\check{E}_{ab} + \frac{\check{E}_{bc}}{2} \right) \\ &= j \sqrt{3} \, \frac{\check{I}_a}{2} \, \{(a + \tfrac{1}{2}) \, Z_1 - (a^2 + \tfrac{1}{2}) \, Z_2\} \\ &= - \tfrac{3}{4} \, \check{I}_a \, (Z_1 + Z_2) \end{aligned}\right\} \tag{153}$$

and therefore,

$$\check{I}_a = - 1\tfrac{1}{3} \, \frac{\check{E}_{ad}}{Z_1 + Z_2} \tag{154}$$

$$P_0 = \tfrac{3}{4}\left\{\frac{w_0 - w_1}{w_1} K_1^2 - \frac{w_0 - w_1}{2\,w_0 - w_1} K_2^2\right\} I_a^2 R_s \qquad (155)$$

$$P_1 + j\,Q_1 = \tfrac{3}{4}\, I_a^2 (Z_1 + Z_2) + P_F \qquad (156)$$

$$\cos\alpha = \frac{P_1}{\sqrt{P_1^2 + Q_1^2}} \qquad (157)$$

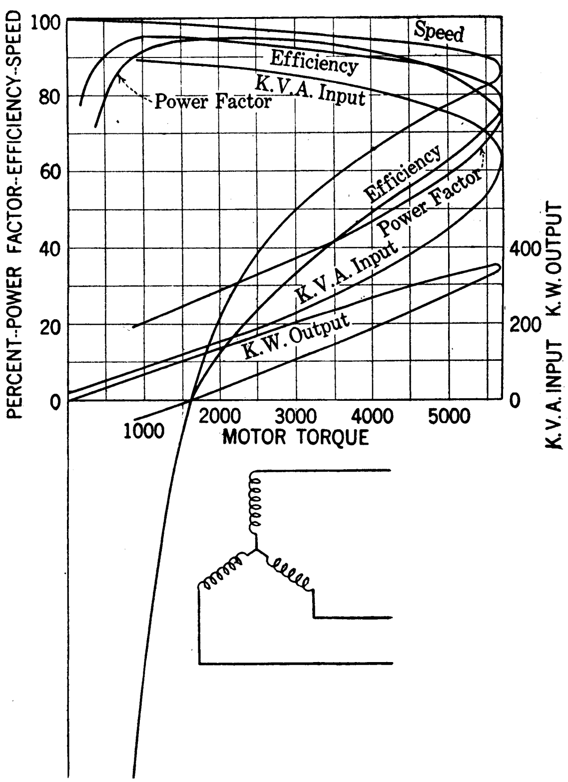

FIG. 6—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—BALANCED THREE-PHASE

Evidently (**155**), (**156**) and (**157**) are identical to (**144**), (**145**) and (**146**) if I_a is equal to $I_b \div \frac{\sqrt{3}}{2}$. This will be the case if the value of $E_{ad} = \frac{\sqrt{3}}{2}$ times that of E_{bc}. The total heating of the motors will be the same in each case but the heating in one phase for Case III will be one-third greater than for Case I.

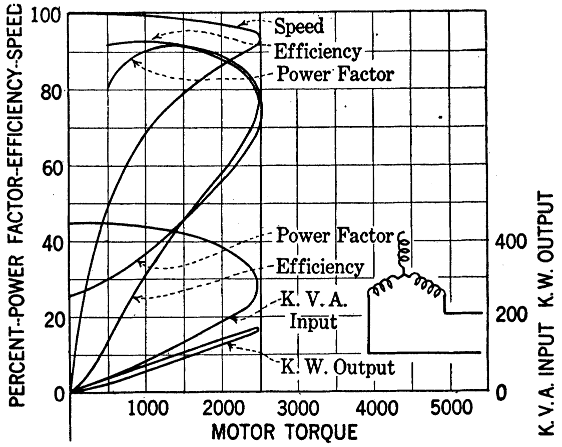

FIG. 7—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—SINGLE-PHASE OPERATION —ONE LEAD OPEN

This method of operation is therefore, as far as total losses, etc. are concerned, the exact counterpart of two polyphase

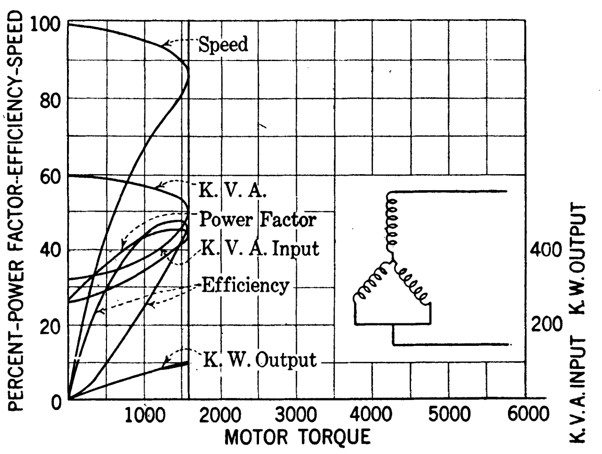

FIG. 8—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—SINGLE-PHASE OPERATION

motors connected in series with shafts mechanically connected, one of which has its phase sequence reversed.

Figs. 6, 7 and 8 show characteristic curves of a three-phase induction motor operating respectively on a symmetrical circuit, according to Case I and according to Case II.

Synchronous Machinery

THE SYMMETRICAL THREE-PHASE GENERATOR OPERATING ON UNSYMMETRICALLY LOADED CIRCUIT

The polyphase salient pole generator is not strictly a symmetrical machine, the exciting winding is not a symmetrical polyphase winding and it therefore sets up unsymmetrical trains of harmonics in exactly the same way as they are set up in an induction motor with unsymmetrical secondary winding. These cases will therefore be taken up later on. A three-phase generator may however be wound with a distributed polyphase winding to serve both as exciting and damper winding and if properly connected will be perfectly symmetrical. Such a machine will differ from an induction motor only in respect to the fact that it operates in synchronism and has internally generated symmetrical e. m. fs. which we will denote by $S^1 \check{E}_{a1}$, $S^2 \check{E}_{a2}$ the negative phase sèquence component being zero; an e. m. f. $S^0 \check{E}_{a0}$ may exist but since in all the connections that will be considered there will be no neutral connection its value may be ignored. If the load impedances be Z_a', Z_b' and Z_c' they may be expressed by

$$Z_{sa}{}^1 = S^0 Z_{a0}{}^1 + S^1 Z_{a1}' + S^2 Z_{a2}'$$

and the equations of the generator will be

$$\left.\begin{aligned}
S^1 \check{E}_{a1} &= S^1 \left\{ \left(R_a + L_a \frac{d}{d\,t} \right) \check{I}_{a1}' + Z_{a0}' I_{a1}' \right. \\
&\qquad \left. + Z_{a2}' I_{a2}' + M \frac{d}{d\,t} e^{j w_0 t} \check{I}_{u1}' \right\} \\
O &= S^2 \left\{ \left(R_a + L_a \frac{d}{d\,t} \right) \check{I}_{a2} + Z_{a0}' I_{a2}' \right. \\
&\qquad \left. + Z_{a1}{}^1 \check{I}_{a1}' + M \frac{d}{d\,t} e^{-j w_0 t} \check{I}_{u2}' \right\} \\
O &= \left(R_u + L_u \frac{d}{d\,t} \right) \check{I}_{u1}' + M \frac{d}{d\,t} e^{-j w_0 t} \check{I}_{a1}' \\
O &= \left(R_u + L_u \frac{d}{d\,t} \right) \check{I}_{u2}' + M \frac{d}{d\,t} e^{j w_0 t} \check{I}_{a2}'
\end{aligned}\right\} \quad (158)$$

The last two equations give

$$\left.\begin{aligned} I_{u1}' &= -\frac{M\dfrac{d}{dt}}{R_u + L_u\dfrac{d}{dt}}\, e^{-jw_0 t}\, \check{I}_{a1}' \\ I_{u2}' &= -\frac{M\dfrac{d}{dt}}{R_u + L_u\dfrac{d}{dt}}\, e^{jw_0 t}\, I_{a2}' \end{aligned}\right\} \quad (\mathbf{159})$$

which on substitution in the first two equations of (**158**) give the equations

$$\left.\begin{aligned} &\left\{ R_a + L_a\frac{d}{dt} - \frac{M^2\dfrac{d}{dt}\left(\dfrac{d}{dt} - j\,w_0\right)}{R_u + L_u\left(\dfrac{d}{dt} - j\,w_0\right)} \right\} \check{I}_{a1}' \\ &\qquad + Z_{a0}^1\, \check{I}_{a1}' + Z_{a2}'\, \check{I}_{a2}' = \check{E}_{a1} \\ &Z_{a1}^1 \check{I}_{a1}' + \left\{ R_a + L_a\frac{d}{dt} \right. \\ &\left. - \frac{M^2\dfrac{d}{dt}\left(\dfrac{d}{dt} + j\,w_0\right)}{R_u + L_u\left(\dfrac{d}{dt} + j\,w_0\right)} \right\} \check{I}_{a2}' + Z_{a0}^1\, \check{I}_{a2}' = 0 \end{aligned}\right\} \quad (\mathbf{160})$$

or if

$$\check{E}_{a1} = E_{a1}\, e^{jw_0 t} \quad (\mathbf{161})$$

the impedances Z_{a0}, Z_{a1}, Z_{a2} become ordinary impedance for an electrical angular velocity w_0 and equations (**160**) become

$$\left.\begin{aligned} &(R_a + j\,w\,L_a + Z_{a0}^1)\,\check{I}_{a1}' + Z_{a2}'\,\check{I}_{a\,2}' = \check{E}_{a1} \\ &Z_{a1}'\,\check{I}_{a1}' + \{Z_{a0}' + (R_a + K_2^2 R_u) + j\,2\,w_0\,(L_a - K_2^2 L_u) \\ &\qquad\qquad - \tfrac{1}{2}\,K_2^2\,R_u\}\,\check{I}_{a2}' = O \end{aligned}\right\} \quad (\mathbf{162})$$

It is apparent that in the generator the impedances

$$R_a + j\,w_0\,L_a = Z_1'$$

and $\{(R_a + K_2^2 R_u) + j\,2\,w_0\,(L_a - K_2^2 L_u) - \tfrac{1}{2}\,K^2 R_u\} = Z_2'$

take the place of Z_1 and Z_2 in the symmetrical induction motor operating on an unsymmetrical circuit, and we may express equation (**162**)

$$\left.\begin{aligned}(Z_{a0}' + Z_1') \check{I}_{a1}' + Z_{a2}' \check{I}_{a2}' &= \check{E}_{a1}\\ Z_a' \check{I}_{a1}' + (Z_{a0}' + Z_2') I_{a2}' &= O\end{aligned}\right\} \quad (\mathbf{163})$$

which gives

$$\check{I}_{a2}' = - \frac{Z_{a1}'}{Z_{a0} + Z_2'} \check{I}_{a1}'$$

$$\check{I}_{a1}' = \frac{\check{E}_{a1}}{(Z_{a0}' + Z_1') - \dfrac{Z_{a1}' Z_{a2}'}{Z_{a0}' + Z_2'}}$$

Or in more symmetrical form

$$\left.\begin{aligned}\check{I}_{a1}' &= \frac{Z_{a0}' + Z_2'}{(Z_{a0}' + Z_1')(Z_{a0}' + Z_2') - Z_{a1}' Z_{a2}'} \check{E}_{a1}\\ \check{I}_{a2}' &= - \frac{Z_{a1}^1}{(Z_{a0}' + Z_1')(Z_{a0}' + Z_2') - Z_{a1}' Z_{a2}'} \check{E}_{a1}\end{aligned}\right\} \quad (\mathbf{164})$$

From (**159**) we have for the damper currents

$$\left.\begin{aligned}\check{I}_{u1}' &= O \text{ if } R_u > O\\ \check{I}_{u2}' &= - \check{K}_2 I_{a2} e^{j 2 u_0 t}\\ &\text{where } \check{K}_2 = j \frac{2 w_0 M}{R_u + j 2 w_0 L_u}\end{aligned}\right\} \quad (\mathbf{165})$$

A particular case of interest is when the load is a Synchronous Motor or Induction Motor with unsymmetrical line impedances in series—Equation (**163**) becomes

$$\left.\begin{aligned}(Z_{a0}' + Z_1' + Z_1) \check{I}_{a1}' + Z_{a2}' I_{a2}' &= \check{E}_{a2}\\ Z_{a1} \check{I}_{a1}' + (Z_{a0}' + Z_2' + Z_2) I_{a2}' &= O\\ I_{a1}^1 &= \frac{Z_{a0}' + Z_2' + Z_2}{(Z_{a0}'+Z_1'+Z_1)(Z_{a0}'+Z_2'+Z_2) - Z_{a1} Z_{a2}} \check{E}_{a1}\\ I_{a2}^1 &= \frac{Z_{a1}}{(Z_{a0}'+Z_1'+Z_1)(Z_{a0}'+Z_2' Z_2) - Z_{a1} Z_{a2}} \check{E}_{a1}\end{aligned}\right\} \quad (\mathbf{166})$$

An important case is that of a generator feeding into a symmetrical motor and an unsymmetrical load. Let the motor currents be

$\check{I}_a$, $\check{I}_b$, $\check{I}_c$, those of the load I_a', I_b', I_c' and the load impedances Z_a', Z_b', Z_c'. The equations of this system will be

$$\left.\begin{aligned} S^1\,\check{E}_{a1} &= S^1\,\{Z_1'\,(\check{I}_{a1} + \check{I}_{a1}') + Z_{a0}'\,\check{I}_{a1}' + Z_{a2}'\,\check{I}_{a2}'\} \\ S^1\,\check{E}_{a1} &= S^1\,\{Z_1'\,(\check{I}_{a1} + \check{I}_{a1}') + Z_1\,\check{I}_{a1}\} \\ S^2\,O &= S^2\,\{Z_2'\,(\check{I}_{a2} + \check{I}_{a2}') + Z_{a0}'\,\check{I}_{a2} + Z_{a1}'\,\check{I}_{a1}'\} \\ S^2\,O &= S^2\,\{Z_2'\,(\check{I}_{a2} + \check{I}_{a2}') + Z_2\,\check{I}_{a2}\} \end{aligned}\right\} \quad (\mathbf{167})$$

Or, omitting the sequence symbols and re-arranging—

$$\left.\begin{aligned} \check{E}_{a1} &= Z_1'\,\check{I}_{a1} + (Z_1' + Z_{a0}')\,\check{I}_{a1}' + Z_{a2}'\,\check{I}_{a2}^1 \\ \check{E}_{a1} &= (Z_1' + Z_1)\,\check{I}_{a1} + Z_1'\,\check{I}_{a1}^1 \\ O &= Z_2'\,\check{I}_{a2} + Z_{a1}'\,\check{I}_{a1}^1 + (Z_2' + Z_{a0}')\,\check{I}_{a2}' \\ O &= (Z_2' + Z_2)\,\check{I}_{a2} + Z_2'\,\check{I}_{a2}' \end{aligned}\right\} \quad (\mathbf{168})$$

These equations can be further simplified as follows:

$$\left.\begin{aligned} O &= (Z_2' + Z_2)\,\check{I}_{a2} + Z_2'\,\check{I}_{a2}' \\ O &= -\,Z_2\,\check{I}_{a2} + Z_{a1}'\,\check{I}_{a1}' + Z_{a0}'\,\check{I}_{a2}' \\ O &= -\,Z_1\,\check{I}_{a1} + Z_{a0}'\,\check{I}_{a0}' + Z_{a2}'\,I_{a2}' \\ \check{E}_{a1} &= (Z_1' + Z_1)\,\check{I}_{a1} + Z_1'\,\check{I}_{a1}' \end{aligned}\right\} \quad (\mathbf{169})$$

A set of simultaneous equations which may be easily solved.

The Single-Phase Generator is an Important Case of the Three-Phase Generator Operated on an Unbalanced Load

Let the impedance of the single-phase load be Z and let us suppose it to be made up of three star connected impedances

$$Z_a' = 3\,Z_x + \frac{Z}{2}$$

$$Z_b' = \frac{Z}{2}$$

$$Z_c' = \frac{Z}{2}$$

the value of Z_x in the limit being infinity. Then we have

$$\left.\begin{aligned} Z_{a0}' &= Z_x + \frac{Z}{2} \\ Z_{a1}' &= Z_x \\ Z_{a2}' &= Z_x \end{aligned}\right\} \qquad \textbf{(170)}$$

Equation (**164**) in the limit when Z_x becomes infinite reduces to

$$\begin{aligned} \check{I}_{a1}' &= \frac{\check{E}_{a1}}{Z + Z_1^1 + Z_2^1} \\ \check{I}_{a2} &= -\frac{\check{E}_{a1}}{Z + Z_1' + Z_2'} \end{aligned} \qquad \textbf{(171)}$$

The single-phase load being across the phase $B\ C$, the single-phase current I will therefore be equal to $\check{I}_c$ or

$$\left.\begin{aligned} \check{I} &= \frac{j\sqrt{3}\,\check{E}_{a1}}{Z + Z_1' + Z_2'} \\ &= \frac{\check{E}_{bc}}{Z + Z_1' + Z_2'} \end{aligned}\right\} \qquad \textbf{(172)}$$

$$\left.\begin{aligned} \check{I}_{u1} &= O \text{ if } R_u > O \\ \check{I}_{u2} &= -j\frac{1}{\sqrt{3}}\check{K}_2\,\check{I}\,e^{jw_0t} \\ \check{I}_{u2} &= -\frac{j\,\check{K}_2}{\sqrt{3}}\,I\,e^{jw_0t} \end{aligned}\right\} \qquad \textbf{(173)}$$

$\check{I}_{u2}$ is double normal frequency

$$\left.\begin{aligned} P_I + j\,Q_I &= 3\,I^2\,Z \\ P_L + j\,Q_L &= 3\,I^2\,(Z_1' + Z_2') \\ (P + j\,Q) + (P_H + j\,Q_H) &= 3\,\check{E}_{bc}\,(\hat{I} + \check{I}) \end{aligned}\right\} \qquad \textbf{(174)}$$

In the case of the generally unbalanced three-phase load

$$\left.\begin{aligned} P_1 + j\,Q_1 &= 3\,\{(I_{a1}^2 + I_{a2}^2)\,Z_{a0}' \\ &\qquad + \check{I}_{a1}\,\hat{I}_{a2}\,Z_{a2}' + \hat{I}_{a1}\,\check{I}_{a2}\,Z_{a1}'\} \\ P_L + j\,Q_L &= 3\,\{I_{a1}^2\,Z_1' + I_{a2}^2\,Z_2'\} \\ (P + j\,Q)\,j\,(P_H + j\,Q_H) &= 3\,\check{E}_{a1}\,(\check{I}_{a2} + \hat{I}_{a2}) \end{aligned}\right| \qquad \textbf{(175)}$$

When the generator has harmonics in its wave form equations (**162**) must be written

$$\left.\begin{aligned} &(R_a + j\,w\,L_a + Z_{a0}')\,\check{I}_{a1}' + Z_{a2}'\,I_{a2}' = \check{E}_{a1} \\ &\quad Z_{a1}'\,I_{a1}' + \{Z_{a0}' + (R_a + K_2^2\,R_u) \\ &\qquad + j\,2\,w\,(L_a - K_2^2\,L_u) - \tfrac{1}{2}\,a^2\,R_u\}\,I_{a2}' = \check{E}_{a2}' \end{aligned}\right\} \qquad \textbf{(176)}$$

Where $\check{E}_{a1}$ is finite, $\check{E}_{a2}$ is zero and vice versa, the frequencies being different in each case, we have therefore a solution for each frequency depending on the phase and amplitude and phase sequence of the e. m. f. of this frequency generated. Of course the values of Z_1' and Z_2' change with each frequency on account of the change in the reactance with frequency, and a value must be taken for w conforming with the frequency of the harmonic under consideration.

Symmetrical Synchronous Motor, Synchronous Condenser, Etc.

As in the case of the generator, the synchronous motor has two impedances, one to the positive phase sequence current of a given frequency and the other to the negative phase sequence current of the same frequency. But, since there is no quantity in the positive phase sequence impedance corresponding to the virtual resistance which indicates mechanical work in an induction motor, its equivalent is furnished by the excitation of the field. Let us denote the e. m. f. due to the field excitation by $S^1\,\check{E}_{a1}'$ assuming it to be for the present a simple harmonic three-phase system. Let P_0 be the output of the motor which will include the windage and iron losses assumed to be constant. Then for the synchronous motor on a balanced circuit of e. m. f. $S^1\,\check{E}_{a1}$ we have

$$S^1\,\check{E}_{a1} = S^1\,\{\check{I}_{a1}\,(R_a' + j\,w\,L_a') + \check{E}_{a1}'\} \qquad \textbf{(177)}$$

$$S^0\,\check{E}_{a1}\,\hat{I}_{a1} = S^0\left\{I_{a1}^2\,(R_a' + j\,w\,L_a') + \frac{P_0}{3} - j\,\frac{Q_0}{3}\right\} \qquad \textbf{(178)}$$

Where Q_0 is the imaginary part of the product, $\check{E}_{a1}'\,\hat{I}_{a1}$. (**178**) reduces to

$$E_{a1}\,I_{a1}\cos\alpha = I_{a1}^2\,R_a^1 + \frac{P_0}{3} \qquad \textbf{(179)}$$

Where $\cos\alpha$ is the required operating power factor. Solving for I_{a1}

$$I_{a1} = \frac{E_{a1} \cos \alpha}{2 R_a^1} \left\{ 1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} \tag{180}$$

$$\check{I}_{a1} = \check{E}_{a1} \frac{\cos \alpha}{2 R_{a1}} \left\{ 1 \pm \sqrt{1 - \frac{4 R_{a1}' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} (\cos \alpha - j \sin \alpha) \tag{181}$$

The apparent impedance of the motor is

$$\frac{2 R_1 \sec \alpha}{1 \pm \sqrt{1 - \frac{4 P_0}{3 E_a^2 \cos^2 \alpha}}} (\cos \alpha + j \sin \alpha) \tag{182}$$

and

$$\check{E}_{a1}^1 = \check{E}_{a1} \left[1 - \frac{\cos \alpha}{2 R_a^1} \left\{ 1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} (\cos \alpha - j \sin \alpha) (R_a' + j w L_a') \right] \tag{183}$$

The same equations apply to the case of the synchronous condenser with the difference that the mechanical work is that required to overcome the iron and windage losses only.

If we take

$$\left. \begin{aligned} \check{E}_{a1} &= E_{a1} (\cos \alpha + j \sin \alpha) \, e^{j w_0 t} = (A_1 + j B_1) \, e^{j w_0 t} \\ \check{E}_{a1}' &= (A_1' + j B_1') \, e^{j w_0 T} \end{aligned} \right\} \tag{184}$$

we have

$$\check{I}_{a1} = \frac{A_1}{2 R_a^1} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) e^{j w_0 t} \tag{185}$$

$$A_1' = \frac{A_1}{2} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) e^{j w_0 t} \tag{186}$$

$$B_1' = \left\{ R_1 - \frac{j w L_a' A_1}{2 R_a'} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) \right\} e^{j w_0 t} \tag{187}$$

Since α may be a positive or negative angle, the sine may be positive or negative for a positive cosine, and therefore the power factor will be leading or lagging accordingly as B_1 is negative or positive respectively. The double signs throughout are due to the fact that for any given load and power factor there are always two theoretically possible running conditions. However, since we are concerned only with that one which will give the max. operating efficiency, that is the condition that gives I_{a1} the lesser value, for a given value of P_0 the equations may be written

$$\left.\begin{aligned} \check{I}_{a1} &= \frac{A_1}{2 R_a^1} \left(1 - \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}}\right) e^{j w_0 t} \\ A_1' &= \frac{A_1}{2} \left(1 + \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}}\right) e^{j w_0 t} \\ B_1' &= \left\{ B_1 - \frac{j w_0 L_a' A_1}{2 R_a'} \left(1 - \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}}\right)\right\} \end{aligned}\right\} \quad (188)$$

And corresponding values for (**180**), (**181**), (**182**) and (**183**) may be obtained by omitting the positive sign in these equations.

Another condition of operation is obtained by inspection of (**180**), due to the fact that I_{a1} must be a real quantity

$$\frac{4 \ R_a' \ P_0}{3 \ E_{a1}^2 \cos^2 \alpha} \text{ must be } > 1 \quad (189)$$

this is the condition of stability. In terms of (**184**) it becomes

$$\frac{4 R_a' P_0}{3 A_1^2} \text{ must be } > 1 \quad (190)$$

The same conditions apply to the synchronous condenser, the total mechanical load in this case being the iron loss and windage and friction losses.

Proceeding now to operation with unbalanced circuits having sine waves the motor also having a sine wave. In addition to equation (**177**) we shall have

$$S^2 \check{E}_{a2} = S^2 Z_2' \check{I}_{a2} \quad (191)$$

The mechanical power delivered through the operation of this negative phase sequence e. m. f. is given by P_N where

$$P_N = - 3 \ I_{a2}^2 \frac{R_a'}{2} \quad (192)$$

this quantity must therefore be subtracted from the value of P_0 in all the equations in which P_0 appears when unbalanced circuits are used in connection with equations (**177**) to (**190**) inclusive. These equations, however, give the conditions for maintaining a given mechanical load and a given power factor in the positive phase sequence component, but in practice what is required is the combined power factor of the whole system, or the conditions to give a certain combined factor while delivering a given mechanical load; this may be obtained as follows:

The negative phase sequence component is a perfectly definite impedance and is independent of the load, and therefore the zero frequency part of the product $E_{a2}\, I_{a2}$ may be set down as

$$\check{E}_{a2}\, \hat{I}_{a2} = \frac{P_2}{3} + j\, \frac{Q_2}{3} \tag{193}$$

we have also for the positive phase sequence power delivered

$$(A_1 + j\, B_1)\; I_{a1} = I_{a1}^2\, R_a^1 + \frac{P_0}{3} - \frac{P_N}{3} + j\,(w\, I_{a1}\, L_a^1 + B_1^1)\; I_{a1} \tag{194}$$

And the power factor is given by

$$\tan\alpha = \frac{I_{a1}\, B_1 + \dfrac{Q_2}{3}}{I_{a1}\, A_1 + \dfrac{P_2}{3}} \tag{195}$$

From (**194**) we have

$$A_1\, I_{a1} = I_{a1}^2\, R_a^1 + \frac{P_0}{3} - \frac{P_N}{3} \tag{196}$$

$$B_1 = w\, I_{a1}\, L_a^1 + B_1^1 \tag{197}$$

$$A_1^2 + B_1^2 = E_{a1}^2 \tag{198}$$

The simplest method of solving these equations is by means of curves. Taking arbitrary values of I_{a1}, B_1 and A_1 are chosen consistent with (**198**) so as to satisfy (**195**), $\dfrac{P_0}{3}$ A_1' and B_1' are then obtained from (**196**) and (**197**). If there are harmonics in the impressed e. m. f. but there are none in the wave form of the machine, the machine will have a definite impedance to the positive and negative phase sequence components of each harmonic, so that there will be a definite amount of mechanical work contributed by each harmonic which must be subtracted from the total work to be done to give the amount of work contributed by the positive phase sequence fundamental component, the equations will be identical to (**193**), (**194**), (**195**), (**196**), (**197**) and (**198**), if we take P_N to mean the total mechanical work done by the harmonics both positive and negative phase sequence and P_2 and Q_2 to represent the products

$$\Sigma\ ({}_n\check{E}_{a1}\quad {}_n\hat{I}_{a1} + {}_n\check{E}_{a2}\quad {}_n\hat{I}_{a2})$$

the zero frequency part only being taken into account.

When harmonics are present both in the impressed wave and in the generated wave, the problem becomes too complicated to treat generally, but specific cases can be worked out without much difficulty.

Phase Converters and Balancers

The phase converter is a machine to transform energy from single-phase or pulsating form to polyphase or non-pulsating form or vice versa to transform energy from polyphase to single-phase. The transformation may not be complete, that is to say, the polyphase system may not be perfectly balanced when supplied from a single-phase source through the medium of a phase converter. Phase converters may be roughly divided into two classes, namely—shunt type and series type.

Induction Motor or Synchronous Condenser Operating as a Phase Converter of the Shunt Type to Supply a Symmetrical Induction Motor or Synchronous Motor

Let Z_1 and Z_2 be the positive and negative phase sequence impedances of the motor, Z_1', Z_2' those of the phase converter. Let $S^1\ \check{E}_{a1}$ and $S^2\ \check{E}_{a2}$ be the positive and negative phase sequence components of the star e. m. f. impressed on the motor as a result of the operation. The single-phase supply will be one side of the delta e. m. f. $S\ \check{E}_{bc}$ which has positive and negative phase sequence components $S^1\ E_{bc1}$ and $S^2\ E_{bc2}$ the single-phase supply being $\check{E}_{bc} = \check{E}_{bc1} + \check{E}_{bc2}$.

The value of Z_2' may be considered fixed for all practical purposes and since in the induction motor phase converter the speed is practically no-load speed, Z' is practically the no-load impedance plus a real part obtained by increasing the real part of the no-load impedance by the ratio of the normal no-load losses to these same losses plus $\frac{1}{2}$ the secondary losses due to the phase converter currents. The latter may be calculated roughly as even a large error in its value will have an inappreciable effect on the actual results. We have therefore

$$\left.\begin{aligned} S^1 \check{E}_{a1} &= - S^1 j \frac{\check{E}_{bc}}{\sqrt{3}} \\ S^2 \check{E}_{a2} &= S^2 j \frac{\check{E}_{bc2}}{\sqrt{3}} \end{aligned}\right\} \quad (199)$$

$$\left.\begin{aligned} S_1^1 \check{I}_{a1}' &= - S^1 j \frac{\check{E}_{bc1}}{\sqrt{3}\, Z_1'} \\ S^1 \check{I}_{a1} &= - S^1 j \frac{\check{E}_{bc1}}{\sqrt{3}\, Z_1} \end{aligned}\right\} \quad (200)$$

$$\left.\begin{aligned} S^2 \check{I}_{a2}' &= S^2 j \frac{\check{E}_{bc2}}{\sqrt{3}\, Z_2'} \\ S^2 \check{I}_{a2} &= S^2 j \frac{\check{E}_{bc2}}{\sqrt{3}\, Z_2} \end{aligned}\right\} \quad (201)$$

In the common lead of motor and converter we have

$$\check{I}_{a1}' + \check{I}_{a2}' + \check{I}_{a1} + \check{I}_{a2} = O \quad (202)$$

or, substituting from (**200**) and (**201**)

$$\check{E}_{bc2} \left(\frac{1}{Z_2'} + \frac{1}{Z_2} \right) = \check{E}_{bc1} \left(\frac{1}{Z_1'} + \frac{1}{Z_1} \right) \quad (203)$$

$$\frac{\check{E}_{bc1}}{\check{E}_{bc2}} = \frac{\frac{1}{Z_2'} + \frac{1}{Z_2}}{\frac{1}{Z_1'} + \frac{1}{Z_1}} \quad (204)$$

$$\check{E}_{bc1} = \frac{\frac{1}{Z_2} + \frac{1}{Z_2'}}{\left(\frac{1}{Z_1} + \frac{1}{Z_1'} \right) + \left(\frac{1}{Z_2} + \frac{1}{Z_2'} \right)} \check{E}_{bc} \quad (205)$$

$$\check{E}_{bc2} = \frac{\frac{1}{Z_1} + \frac{1}{Z_1'}}{\left(\frac{1}{Z_1} + \frac{1}{Z_1'} \right) + \left(\frac{1}{Z_2} + \frac{1}{Z_2'} \right)} \check{E}_{bc} \quad (206)$$

which give the complete solution for all the quantities required with the aid of equations (**200**) and (**201**). For the supply current $\check{I}$

$$\left.\begin{aligned} \check{I} &= \check{I}_{bc1} + \check{I}_{bc2} + I_{bc1}' + I_{bc2}' \\ S\,\check{I}_{bc} &= S^1\,\check{I}_{bc1} + S^2\,\check{I}_{bc2} \\ S\,\check{E}_{bc} &= S^1\,\check{E}_{bc1} + S^2\,E_{bc2} \end{aligned}\right\} \quad (\mathbf{207})$$

$$P_1 + j\,Q_1 = \check{E}_{bc}\,\hat{I} \quad (\mathbf{208})$$

In order to obtain a perfect balance we may consider the addition of an e. m. f. $S^2\,j\,\frac{E_{x2}}{\sqrt{3}}$ in series with the phase converter whose value must be a function of the load and the phase converter impedances, and therefore equation (**201**) will be replaced by

$$S^2\,\check{I}_{a2}' = S^2\left(j\,\frac{\check{E}_{bc2}}{\sqrt{3}\,Z_2'} + j\,\frac{\check{E}_{x2}}{\sqrt{3}\,Z_2'}\right) \quad (\mathbf{209})$$

$$S^2\,\check{I}_{a2} = S^2\,j\,\frac{\check{E}_{bc}}{\sqrt{3}\,Z_2}$$

and since the balance is perfect $\check{E}_{bc2}$ is zero, and therefore

$$S^2\,j\,\frac{\check{E}_{x2}}{\sqrt{3}} = S^2\,Z_2'\,\check{I}_{a2}' \quad (\mathbf{210})$$

An e. m. f. equal and of opposite phase to the negative phase sequence drop through the phase converter is required to produce a perfect balance.

Carrying out the solution in the same manner as in the imperfect converter, we obtain

$$\check{E}_{bc2} = \frac{\frac{1}{Z_1} + \frac{1}{Z_1'}}{\frac{1}{Z_2} + \frac{1}{Z_2'}}\,E_{bc} - \frac{\frac{1}{Z_2'}}{\frac{1}{Z_2} + \frac{1}{Z_2'}}\,\check{E}_{x2} \quad (\mathbf{211})$$

and since $\check{E}_{bc2}$ is zero and $\check{E}_{bc1} = \check{E}_{bc}$ the single-phase impressed e. m. f., we obtain

$$\check{E}_{x2} = Z_2'\left(\frac{1}{Z_1} + \frac{1}{Z_1^1}\right)\check{E}_{bc} \quad (\mathbf{212})$$

and therefore from (**210**)

$$S^2\,\check{I}_{a2}' = S^2\,j\left(\frac{1}{Z_1} + \frac{1}{Z_1'}\right)\frac{\check{E}_{bc}}{\sqrt{3}} \quad (\mathbf{213})$$

$$S^1 \check{I}_{a1}{}^{1} = - S^1 j \frac{\check{E}_{bc}}{\sqrt{3}\, Z_1{}'} \tag{214}$$

$$S^2 \check{I}_{a2} = O \tag{215}$$

$$S^1 \check{I}_{a1} = - S^1 j \frac{\check{E}_{bc}}{\sqrt{3}\; Z_1} \tag{216}$$

Figs. 9, 10, 11 and 12 are vector diagrams of some of the principal compensated shunt type phase converters. There will be no

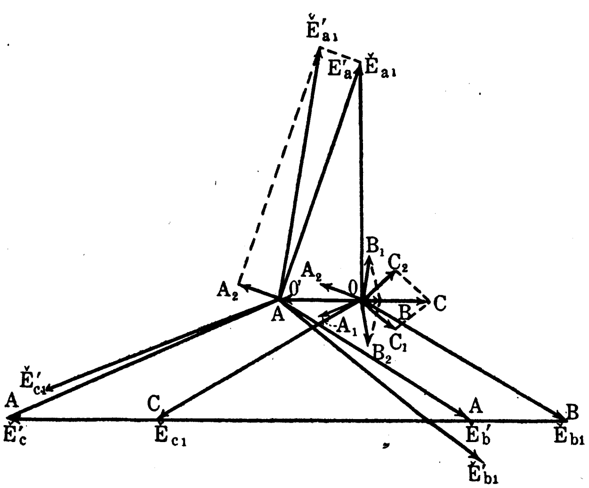

FIG. 9—VECTOR DIAGRAM OF SHUNT-TYPE PHASE CONVERTER OPERATED FROM TRANSFORMER SO AS TO DELIVER BALANCED CURRENTS

Terminal voltages of phase converter $S\check{E}^1{}_a$

Terminal voltages of motor $S^1\check{E}_{a1}$

Nega ive phase sequence.e.m.fs. in phase converter $S^2\ (OA_2)$

difficulty in following out these diagrams if the principles of this paper have been grasped.

The Phase Balancer is a device to maintain symmetry of e. m. fs. at a given point in a polyphase system. It may consist of an induction motor or synchronous condenser with an auxiliary machine connected in series to supply an e. m. f. always proportional to the product of the negative phase sequence current passing through the machine and the negative phase sequence impedance of the balancer. It therefore has the effect of annulling the impedance of the machine to the flow of negative phase sequence current. Thus, in a symmetrical polyphase network, where we have an unbalanced system of currents due to certain conditions

$$S\,\check{I}_a = S^1\,\check{I}_{a1} + S^2\,\check{I}_{a2} \qquad \textbf{(217)}$$

If a balancer be placed at the proper point the component $S^2\,\check{I}_{a2}$ will circulate between the loads and the phase balancer, the other component $S^1\,I_{a1}$ being furnished from the power house. On the other hand, if there be a dissymmetry in the impedance of the system up to the phase balancer, the latter will draw a negative phase sequence current sufficient to counteract the unbalance

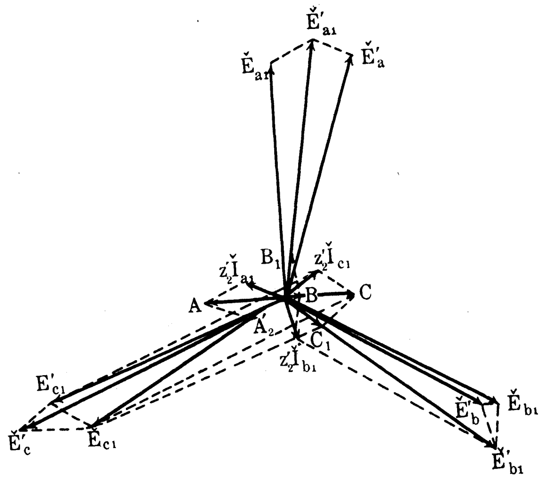

FIG. 10—VECTOR DIAGRAM SHOWING RELATIONS BETWEEN MOTOR TERMINAL E.M.F'S., CONVERTER TERMINAL E.M.FS., AND SYMMETRICAL GENERATED E.M.F'S., SAME CONNECTION AS FOR FIG. 9.

Negative phase sequence drops in phase converter $S^2 Z_2{}^1 \check{I}_{a1}$
Conjugate positive phase sequence e.m.fs. $S^1(ABC)$

due to any symmetrical load by causing the proper amount of negative phase sequence current to flow to produce a balance.

The balancer may be made inherently self-balancing by inserting in series with it a machine which is self-exciting and is able to furnish an e. m. f. equal to the negative phase sequence impedance drop. The combination thus has zero impedance to negative phase sequence currents. If in the neighborhood of a phase balancer the loads have impedances

$$S\,Z_a = S^0\,Z_{a0} + S^1\,Z_{a1} + S^2\,Z_{a2}$$

The equations of the system are

$$\left.\begin{aligned} S^1 \check{E}_{a1} &= S^1 Z_{a0} \check{I}_{a1} + S^1 Z_{a2} \check{I}_{a2} \\ S^2 E_{a2} &= O = S^2 Z_{a0} \check{I}_{a2} + S^2 Z_{a1} \check{I}_{a1} \end{aligned}\right\} \quad (\mathbf{218})$$

The currents in the phase converter are

$$- S^2 \check{I}_{a2} \text{ and } S^1 \frac{\check{E}_{a1}}{Z_1^1}$$

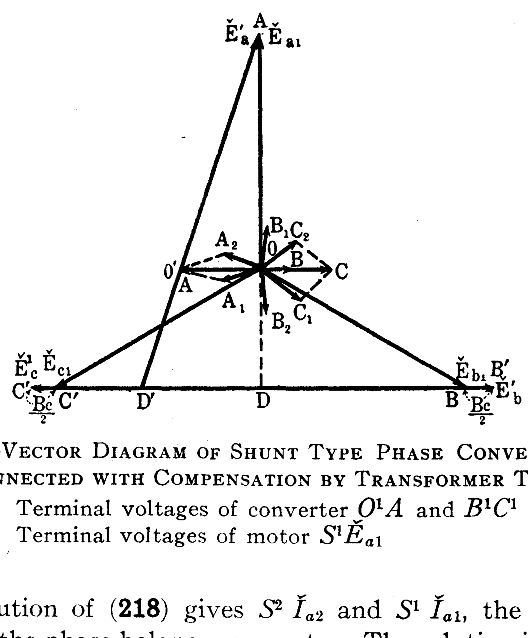

FIG. 11—VECTOR DIAGRAM OF SHUNT TYPE PHASE CONVERTER SCOTT CONNECTED WITH COMPENSATION BY TRANSFORMER TAPS
Terminal voltages of converter O^1A and B^1C^1
Terminal voltages of motor $S^1\check{E}_{a1}$

The solution of (**218**) gives $S^2 \check{I}_{a2}$ and $S^1 \check{I}_{a1}$, the former of which are the phase balancer currents. The solution is

$$\left.\begin{aligned} \check{I}_{a1} &= \frac{Z_{a0}}{Z_{a0}^2 - Z_{a1} Z_{a2}} \check{E}_{a1} \\ \check{I}_{a2} &= - \frac{Z_{a1}}{Z_{a0}^2 - Z_{a1} Z_{a2}} \check{E}_{a1} \end{aligned}\right\} \quad (\mathbf{219})$$

The phase balancer is a voltage balancer and will maintain balanced e. m. f. for any condition of impedance, and if the impedance of the mains is unsymmetrical it will draw a sufficient amount of wattless negative phase sequence current through these mains to produce an e. m. f. balance at its terminals. Hence the complete solution requires consideration of all the connections in the network between the supply point and the balancer. Two equations for each mesh and connection are required, one of the positive phase sequence e. m. fs. and the other of the negative phase sequence e. m. f., and these equations may be solved in the usual way.

Series Phase Converter. In discussing the various reaction in rotating machines we have made use of the terms "positive phase sequence impedance" and "negative phase sequence impedance." These terms are definite enough when dealing with relations between machines whose generated e. m. fs. all have the

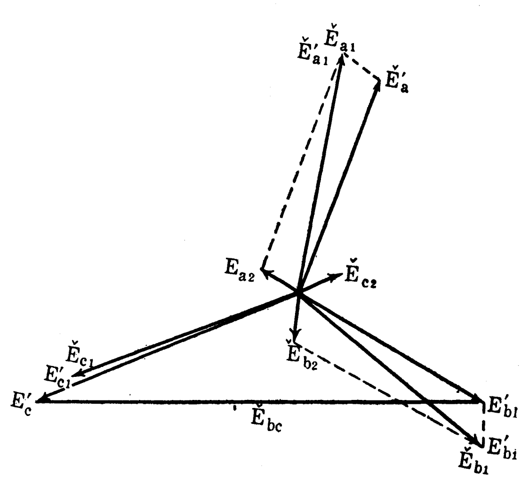

FIG. 12—VECTOR DIAGRAM OF SHUNT-TYPE PHASE CONVERTER WITH AUXILIARY ROTATING COMPENSATOR TO EFFECT A PERFECT BALANCE

Terminal voltages of phase converter $S\ \check{E}_a{}^1$
Terminal voltages of motor $S^1 \check{E}_{a1}$
Terminal voltages of compensator $S^2 \check{E}_{a2}$

same phase sequence, but require further definition when we are dealing with relations between machines whose e. m. fs. have different phase sequence. We shall retain the symbols Z_1 and Z_2 for the values of the positive and negative phase sequence impedances, depending upon the sequence symbol S to define whether these impedances apply to a negative or positive phase sequence current. Thus, the phase sequence of the currents and e. m. f. will be defined by the apparatus supplying and receiving power and the impedances of the transmitting apparatus will be defined in relation to these currents. As an example a motor series connected in counter phase sequence relation in a circuit and driven in a positive direction will have impedances

$$\begin{aligned} &\text{positive phase sequence } Z_2 \\ &\text{negative phase sequence } Z_1 \end{aligned} \qquad \textbf{(220)}$$

Where an auxiliary machine is defined as being of negative phase sequence relation to other machines, it will have impedances as given above to the positive and negative phase sequence currents passing through the other machines.

A single-phase transformer winding tapped at the middle point

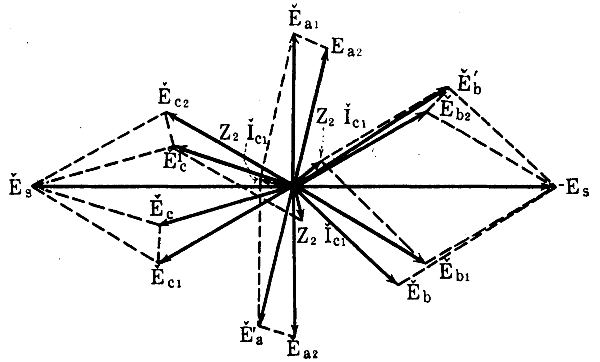

FIG. 13—VECTOR DIAGRAM OF SERIES-TYPE CONVERTER.
NO LOAD E.M.F'S. ACROSS MOTOR TERMINALS $S_1\check{E}_{a1}$
NO LOAD E.M.F'S. ACROSS CONVERTER TERMINALS $S^2\check{E}_{a2}$
SINGLE-PHASE E.M.F'S. $2\check{E}_s$
E.M.F. ACROSS TERMINAL OF MOTOR UNDER LOAD $\check{E}_a\check{E}_a\check{E}_c$
E.M.F. ACROSS TERMINAL OF CONVERTER UNDER LOAD $\check{E}_a\check{E}_b\check{E}_c$

may be regarded as an unbalanced three-phase system where

$$\check{E}_a = O \quad \check{E}_b = +\check{E}_s \quad \check{E}_c = -\check{E}_s$$

$2\,\check{E}_s$ being the single-phase e. m. f The system may be represented by the equation

$$S\,\check{E}_a = S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2}$$

$$\left.\begin{aligned} \text{where } \check{E}_{a1} &= j\,\frac{\check{E}_s}{\sqrt{3}} \\ \check{E}_{a2} &= -j\,\frac{\check{E}_s}{\sqrt{3}} \end{aligned}\right\} \qquad \textbf{(221)}$$

If, therefore between the single-phase source of power and the load we interpose a polyphase machine with e. m. f. $- S^2 \check{E}_{a2}$, we shall have at the load terminals the e. m. f. $S^1 \check{E}_{a1}$. If we use an induction type phase converter it will have impedances to motor currents as follows

$$\left.\begin{array}{l} \text{To positive phase sequence } Z_2' \\ \text{To negative phase sequence } Z_1' \end{array}\right\} \quad (222)$$

we therefore have the relations

$$S^1 \check{E}_{a1} = S^1 \check{I}_{a1} (Z_1 + Z_2') \quad (223)$$

$$S^2 \check{E}_{a2} = S^2 \check{I}_{a2} (Z_2 + Z_1') \quad (224)$$

If the converter is doing no mechanical work, Z_1' is large compared with Z_2' or Z_2, and therefore the component of negative phase sequence is small in the motor. The value of Z_1' depends upon the slip of the phase converter which will depend on the mechanical load it carries as well as on the load carried by the motors. Approximately the load currents due to the motors produce the equivalent at the phase converter of a mechanical load equal to one-half the rotor loss of the phase converter due to these load currents. Substituting the values given in (**221**) for $S^1 \check{E}_{a1}$ and $S^2 \check{E}_{a2}$, we obtain

$$\left.\begin{array}{rl} S^1 j \dfrac{\check{E}_s}{\sqrt{3}} &= S^1 \check{I}_{a1} (Z_1 + Z_2') \\[2ex] - S^2 j \dfrac{E_s}{\sqrt{3}} &= S^2 \check{I}_{a2} (Z_2 + Z_1') \end{array}\right\} \quad (225)$$

$$\begin{array}{rl} S^1 \check{I}_{a1} &= S^1 j \dfrac{\check{E}_s}{\sqrt{3}(Z_1 + Z_2')} \\[2ex] S^2 \check{I}_{a2} &= - S^2 j \dfrac{\check{E}_s}{\sqrt{3}(Z_2 + Z_1')} \end{array} \quad (226)$$

If instead of an induction type phase converter a synchronous phase converter is used an e. m. f. of negative phase sequence $S^2 \check{E}_{a2}$ the generated e. m. f. of the phase converter must be introduced in equations (**224**) *and* (**225**) *and the value and phase of these e. m. fs. will depend upon the load on the phase converter shaft as well as the load carried by the motors.* The equations will be

$$S^1 \check{E}_{a1} = S^1 \check{I}_{a1} (Z_1 + Z_2') \tag{227}$$

$$S^2 \check{E}_{a2} = S^2 \check{I}_{a2} (Z_2 + Z_1') + S^2 \check{E}_{a2}' \tag{228}$$

or

$$\left.\begin{aligned} S^1 j \frac{\check{E}_s}{\sqrt{3}} &= S^1 \check{I}_{a1} (Z_1 + Z_2') \\ - S^2 j \frac{\check{E}_s}{\sqrt{3}} &= S^2 \check{I}_{a2} (Z_2 + Z_1') + S^2 \check{E}_{a2}' \end{aligned}\right\} \tag{229}$$

The last member of equations (**229**) is the equation of a synchronous condenser. Assuming its windage, iron loss and increased losses due to secondary reactions to be P_0, we have by equation (**160**) of the Section on Synchronous Motors

$$\frac{E_s}{\sqrt{3}} I_{a2} \cos \alpha = I_{a2}^2 (R_2 + R_1') + \frac{P_0}{3} \tag{230}$$

Let

$$\check{I}_{a2} = a_2 + j\, b_2 \tag{231}$$

then (**230**) becomes

$$\frac{E_s}{\sqrt{3}} a_2 = (a_2^2 + b_2^2)(R_2 + R_1') + \frac{P_0}{3} \tag{232}$$

Of the two quantities a_2 and b_2, b_2 alone is arbitrary and depends upon the excitation, a_2 will depend upon the value of b_2 and also upon the losses. Solving therefore for a_2 in terms of b_2, we have

$$a_2 = \frac{E_s}{2\sqrt{3}(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4(R_2 + R_1') \{3 b_2^2 (R_2 + R_1') + P_0\}}{E_s^2}} \right\} \tag{233}$$

Since b_2 is arbitrary we may now determine $\cos \alpha_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2}}$ and the value of $\check{I}_{a2}$ in terms of the impressed e. m. f. will be by (**181**) of Section on Synchronous Motors

$$S^2 \check{I}_{a2} = - S^2 \left[j \frac{\check{E}_s}{\sqrt{3}} \frac{\cos \alpha_2}{2(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4(R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}} \right\} \epsilon^{j\alpha} \right] \tag{234}$$

The effective value of $\check{I}_{a2}$ in terms of the effective value of $\check{E}_s$ will then be

$$I_{a2} = \frac{E_s}{\sqrt{3}} \frac{\cos \alpha_2}{2\,(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}} \right\} \tag{235}$$

and since the component of the e.m.f. generated in phase with the current is determined only by the magnitude of $\check{I}_{a2}$ and the motor losses, if we define its value by A_2' the quadrature component being B_2' we shall have

$$A_2' = \frac{E_s}{\sqrt{3}} \frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}} \right) \tag{236}$$

and

$$B_2' = - \frac{E_s}{\sqrt{3}} \sin \alpha_2 - \frac{w\,(L_2 + L_2')}{A_2^1} \tag{237}$$

$$= - \frac{E_s}{\sqrt{3}} \left\{ \sin \alpha_2 + \frac{3\,w\,(L_2 + L_1')}{P_0} \cdot \frac{\cos \alpha_2}{2\,(R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}} \right) \right\} \tag{238}$$

and therefore we have

$$\check{E}_2' = - j \frac{E_s}{\sqrt{3}} \left[\frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}} \right) - j \left\{ \sin \alpha_2 + \frac{3\,w\,(L_2 + L_1') \cos \alpha_2}{2\,P_0\,(R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}} \right) \right\} \right] \tag{239}$$

The impedance of the phase converter to the flow of negative phase sequence current is

$$\frac{2\,(R_2 + R_1') \sec \alpha}{1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha}}} \tag{240}$$

The balance will be at its best when $\check{I}_{a2}$ is a minimum with $\cos \alpha_2$ as the independent variable. This will be the case when $\cos \alpha_2$ is unity; that is to say when b_2 is zero.

Recapitulating the results given above, we have for the general case taking the single-phase e. m. f. $\check{E}_s$ as reference

$$S^1 \check{I}_{a1} = S^1 j \frac{\check{E}_s}{\sqrt{3}\,(Z_1 + Z_2')} \tag{241}$$

$$S^2 \check{I}_{a2} = -j\,(a_2 + j\,b_2) \tag{242}$$

where b_2 is arbitrary and

$$a_2 = \frac{E_s}{2\sqrt{3}\,(R_2 + R_1')}\left\{1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,\{3\,b_2^2\,(R_2 + R_1') + P_0\}}{E_s^2}}\right\} \tag{243}$$

Since b_2 is arbitrary $\cos \alpha_2$ is determined by

$$\cos \alpha_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2}} \tag{244}$$

we may express $\check{I}_{a2}$ in terms of $\check{E}_s$ by

$$S^2 I_{a2} = -S^2 j \frac{\check{E}_s}{\sqrt{3}} \frac{\cos \alpha_2}{2\,(R_2 + R_1')}\left\{1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right\} e^{j\alpha_2} \tag{245}$$

The effective value of $\check{I}_{a2}$ will be

$$I_{a2} = \sqrt{a_2^2 + b_2^2} = \frac{E_s}{\sqrt{3}} \frac{\cos \alpha_2}{2\,(R_2 + Q_1')}\left\{1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right\} \tag{246}$$

If A_2' and B_2' are components of $\check{E}_{a2}'$ these being the generated e. m. f. in phase and in quadrature with the current $\check{I}_{a2}$ we shall have

$$\check{E}_{a2}' = -j\,(A_2' + j\,B_2') \tag{247}$$

and A_2' and B_2' will have the following values

$$A_2' = \frac{E_s}{\sqrt{3}} \; \frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right) \tag{248}$$

$$B_2^1 = -\frac{E_s}{\sqrt{3}} \left\{ \sin \alpha_2 + \frac{3\,w\,(L_2 + L_1')}{P_0} \cdot \frac{\cos \alpha_2}{2\,(R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4\,(R_2 + Q_1')\,P_0}{E_s^2 \cos \alpha^2}}\right)\right\} \tag{249}$$

and $\check{E}_{a2}'$ expressed in terms of $\check{E}_s$ becomes

$$\check{E}_2' = -j\,\frac{\check{E}_s}{\sqrt{3}} \left[\frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right) - j \left\{ \sin \alpha_2 + \frac{3\,w\,(L_2 + L')\cos \alpha_2}{2\,P_0\,(R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right)\right\}\right] \tag{250}$$

The effective impedance of the phase converter to the flow of negative phase sequence currents is

$$\frac{2\,(R_2 + R_1')\sec \alpha_2}{1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha}}} \,(\cos \alpha_2 - j \sin \alpha_2) \tag{251}$$

or

$$\frac{E_s^2}{P_0} \cdot \frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2 \alpha_2}}\right) e^{-j\alpha_2} \tag{252}$$

In the above equations $\cos \alpha_2$ is arbitrary or b_2 may be considered arbitrary and $\cos \alpha_2$ will then be determined.

Minimum Unbalance is obtained when $\cos \alpha_2$ is made unity or when b_2 is made zero in equations (**241**) and (**252**).

Perfect Balance is obtained by driving the phase converter mechanically so as to supply the mechanical power P_0 from a separate or symmetrical source. Under this condition a_2 and b_2 both become zero when $\cos \alpha_2$ is unity. The only equation of the system is then (**241**).

Currents and Power Factor in the Single-Phase Supply Circuit. The e. m. f. is $2\,\check{E}_s$ and the current supplied is

$$\check{I}_s = \frac{\check{I}_b - \check{I}_c}{2}$$

$$= \frac{\check{I}_{b1} - \check{I}_{c1}}{2} + \frac{\check{I}_{b2} - \check{I}_{c2}}{2} \tag{253}$$

If we take

$$S^1\,\check{I}_{a1} = S^1\,j\,(a_1 - j\,b_1) \tag{254}$$

$$\frac{\check{I}_{b1} - \check{I}_{c1}}{2} = \frac{\sqrt{3}}{2}\,(a_1 - j\,b_1) \tag{255}$$

Similarly, since under the same conditions

$$S^2\,\check{I}_{a2} = -\,S^2\,j\,(a_2 + j\,b_2) \tag{256}$$

$$\frac{\check{I}_{b2} - \check{I}_{c2}}{2} = \frac{\sqrt{3}}{2}\,(a_2 + j\,b_2) \tag{257}$$

and therefore

$$\check{I}_s = \frac{\sqrt{3}}{2}\,\{(a_1 + a_2) - j\,(b_1 - b_2)\} \tag{258}$$

where a_1, b_1, a_2, b_2 are to be obtained by means of equations (**243**) to (**254**). The single-phase power factor is given by

$$\tan\theta = \frac{b_1 - b_2}{a_1 + a_2} \tag{259}$$

of these quantities a_2 is usually the smallest and its value may be obtained approximately by assigning to b_2 a value which will make the ratio $\frac{b_1 - b_2}{a_1}$ equal to $\tan\theta$, and obtaining the corresponding value of a_2 by (**242**), the value of b_2 may then be recalculated from (**259**) by substituting the tentative value obtained for a_2. This procedure may be repeated until sufficient accuracy has been obtained.

Single Phase Power Factor in Shunt Type Phase Converter

The simplest procedure is to obtain a curve of admittances for varying excitation of the converter and plot the power factor obtained by varying the admittance with a fixed load. The true and wattless power is obtained easily by means of (**208**) whether the system is balanced or unbalanced.

Figs. 14, 15, 16 and 17 are vector diagrams of several methods of using phase converters to supply a balanced 3-phase e. m. f. to a symmetrical load such as an induction motor. The diagram are all based on a main machine having the same negative phase sequence impedance and the system in each case is

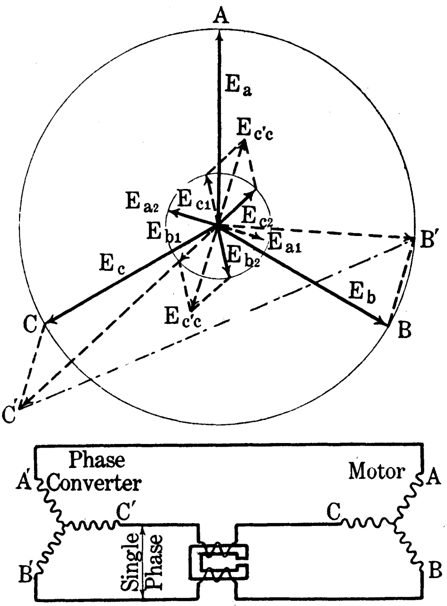

FIG. 14

SINGLE -PHASE IMPRESSED E.M.F. $= B'C'$

MOTOR E.M.F. $= BC$

NEGATIVE PHASE SEQUENCE E.M.FS. $\check{E}_{a2}\check{E}_{b2}\check{E}_{c2}$

CONJUGATE POSITIVE PHASE SEQUENCE E.M.FS. $\check{E}_{a1}\check{E}_{t1}\check{E}_{c1}$

PHASE CONVERTER TERMINAL E.M.F. $AB'C'$

delivering the same amount of power at the same voltage and 3-phase power factor without supplying any wattless power. It will be noted that the scheme Fig. 14 has the lowest single phase power factor, Fig. 16 the highest and the rest arcing alike. It may be remarked, however, that with the shunt type schemes adjustments can be made for power factor correction which will result also in better regulation.

APPENDIX I

Cylindrical Fields in Fourier Harmonics

When we have a diametrical coil around a cylinder concentric with another cylinder which forms the return magnetic path, and the length of the gap is uniform and the coil dimension very small, the field across the gap takes the form of a square topped

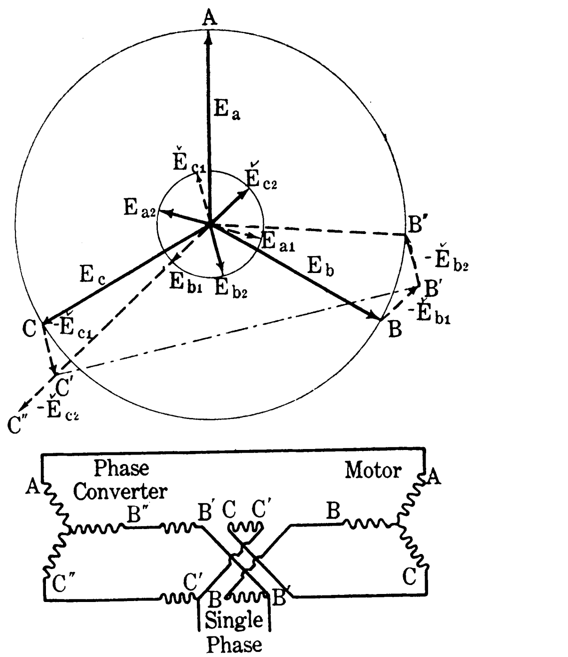

FIG. 15

SINGLE PHASE IMPRESSED E.M.F. = $B'C'$
MOTOR E.M.F. = BC
PHASE CONVERTER E.M.F. = $B''C''$
NEGATIVE PHASE SEQUENCE E.M.F $\check{E}_{a2}\check{E}_{b2}\check{E}_{c2}$
CONJUGATE POSITIVE PHASE SEQUENCE E.M.F. $\check{E}_a{}^1 E_{b1} E_{c1}$
PHASE CONVERTER TERMINAL E.M.F. $AB''C''$

wave, which may be expressed in the form of a Fourier series with the plane of symmetry of the coil as reference plane, and its Fourier expansion is

$$\mathcal{B} = \frac{4\,B}{\pi}\left(\cos\theta - \tfrac{1}{3}\cos 3\,\theta + \frac{1}{S}\cos S\,\theta - \,.\,.+\,.\,.\right) \qquad (1)$$

where B is the average induction in the air gap.

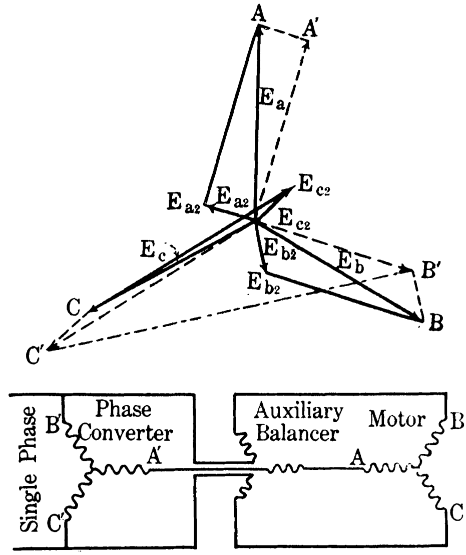

FIG. 16—PHASE CONVERTER WITH AUXILIARY BALANCER.

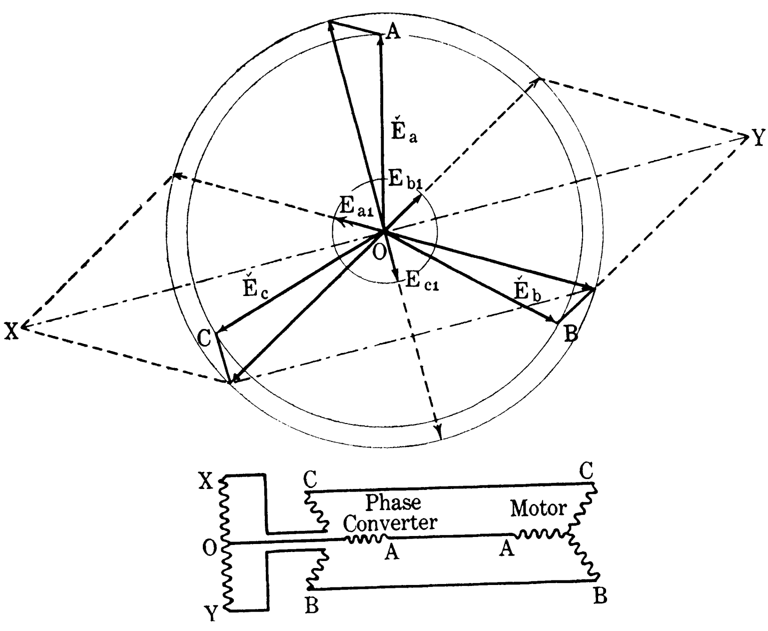

FIG. 17

SINGLE PHASE IMPRESSED E.M.F. = XY

MOTOR E.M.F. = BC

THERE IS A 2 TO 1 TRANSFORMATION OF E.M.F. FROM SINGLE-PHASE TO THREE-PHASE IN THIS CONNECTION

With pitch less than π the curve will have a different form, the amplitude being greater on one side of the plane of the coils than on the other, the areas of each wave will remain the same and second harmonic terms will appear. Let $2\,m_0\,\pi$ be the new pitch then the average amplitude of the induction will be the same as before, namely B, and the value on one side of the coil will be $2\,(1 - m_0)B$ and on the other side $2\,m_0\,B$ so that the total flux will be the same on either side. To obtain the values of the coefficients we have

$$2\,(1 - m_0 B \int_0^{m_0 \pi} \cos n\,\theta\,d\,\theta + 2\,m_0\,B \int_{m_0 \pi}^{2\pi} \cos n\,\theta\,d\,\theta = \frac{\pi}{2}\,A_n$$

$$2\,(1 - m_0)\,B \left[\frac{1}{n} \sin n\,\theta\right]_0^{m_0 \pi} - 2\,m_0\,B \left[\frac{1}{n} \sin n\,\theta\right]_{m_0 \pi}^{2\pi} = \frac{\pi}{2}\,A_n$$

$$A_n = \frac{4\,B}{\pi} \left\{ \frac{(1\ \ m_0) + m_0}{n} \sin n\,m_0\,\pi \right\}$$

$$A_n = \frac{4\,B}{\pi} \left(\frac{1}{n} \sin n\,m_0\,\pi_1 \right) \qquad (\mathbf{2})$$

Let $2\,m_0\,\pi = \frac{2}{3}\,\pi$, then $(1 - m_0)\,\pi = \frac{2}{3}\,\pi$ and

$$\mathcal{B} = \frac{2\,\sqrt{3}\,B}{\pi} \left(\cos\theta + \frac{1}{2} \cos 2\,\theta - \frac{1}{4} \cos 4\,\theta - \frac{1}{5} \cos 5\,\theta \right.$$

$$\left. + \frac{1}{7} \cos 7\,\theta + \frac{1}{8} \cos 8\,\theta - \frac{1}{10} \cos 10\,\theta\ .\ .\ . \right) \qquad (\mathbf{3})$$

A general expression for $\mathcal{B}$ where B is the average of the positive and negative, maximum value for any pitch coil would be

$$\mathcal{B} = \frac{4\,B}{\pi}\,\Sigma \left(\frac{1}{n} \sin n\,m_0\,\pi \cos n\,\theta \right) \qquad (\mathbf{4})$$

and includes all possible coil pitches. If the number of teeth in a pole pitch be n_T; in addition to the average induction as indicated by (**4**), there will also be a tooth ripple of flux, the maximum value of which will depend upon the average value of the induction at each point. The value of m_0 must be a fraction having n_T as denominator and an integral numerator. The value of the integral numerator is therefore always $m_0\, n_T$. The correct value for the max. induction will therefore be

$$\mathfrak{B}_m = \left\{ \frac{4\,B}{\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0\, \pi \cos n\, \theta \right) \right\} (1$$

$$- (-\,1)^{m_0 n_T}\, K_T \cos n_T\, \theta) \qquad (5)$$

where K_T is the ratio of the average to the min. air gap. "m_0" must always be chosen so that $m_0\, n_T$ is an integer.

If the length of the average effective air gap in centimeters be d the value of B is given by

$$B = \frac{4\,\pi}{10} \; \frac{I\,N}{2\,d} \text{ gauss}$$

where I is the maximum value of the current in the coil and N is the number of turns. If d is given in inches we may write

$$B = \frac{4\,\pi}{10} \; \frac{I\,N}{2\,d} \times 2.54 \quad \text{maxwells per square inch.}$$

If we integrate (5) between the limits $(\theta - m_0\, \pi)$ and $(\theta + m_0\, \pi)$ we shall have the total flux φ through the coil

$$\varphi = \frac{4\,B\,r\,l}{\pi} \int_{\theta - m_0\,\pi}^{\theta + m_0\,\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0\, \pi \cos n\, \theta \right) d\,\theta$$

$$- \frac{4\,B\,r\,l}{\pi} (-\,1)^{m_0\, n_T} \int_{\theta - m_0\,u}^{\theta + m_0\,\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0 \,(\cos n\, \theta \right) K_T \cos n_T\, \theta\, d\, 0$$

$$= \frac{4\,B\,re}{\pi} \left[\frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, \theta \right]_{\theta - m_0\,\pi}^{\theta + m_0\,\pi}$$

$$- \frac{4\,B\,r\,l}{\pi} (-\,1)^{m_0\, n_T}\, K_T\, \Sigma \frac{1}{n}\, \theta\, n\, m_0\, n\, \pi \left[\frac{\sin (n - n_T)\, \theta}{2\,(n - n_T)} + \frac{\sin (n + n_T\, \theta)}{2\,(n + n_T)} \right]_{\theta - m_0\,\pi}^{\theta + m_0\,\pi} \qquad (6)$$

The second expression is zero for all values of θ which are integral multiples of the tooth pitch angle, so long as $m_0\, n$ is also an integer and therefore it is zero for all mutual inductive relations of similar coils on a symmetrical toothed core we therefore have:

The induction through a coil displaced an angle θ from the axis of a similar coil carrying a current giving a mean induction B both coils being wound on the same symmetrical toothed core is

$$\varphi = \frac{8\,B\,r\,l}{\pi}\,\Sigma\left(\frac{1}{n^2}\sin^2 n\,m_0\,\pi\cos n\,\theta\right) \qquad (7)$$

The second term in equation (**6**) also becomes zero when n_T becomes infinite independent of the value of θ. We may therefore safely make use of an imaginary uniformly distributed winding when considering self and mutual impedances. It will also be shown later on that with certain groupings of windings the second term may be reduced to zero for every value of θ.

If N_1 be the total number of complete loops in one complete pole pitch, we may take $\frac{N_1}{2\,\pi}$ as the density of winding per unit angle of the complete pole pitch. The mutual induction per turn in a coil angularly displaced an angle θ from another coil of winding density $\frac{N_1}{2\,\pi}$ with an effective total air gap $2\,d$ and with windings subtending an angle $2\,m_1\,\pi$ is given by

$$M_1 = \frac{8\,N_1\,r\,l}{10^9\,\pi\,d}\int_{-m_1\pi}^{+m_1\pi}\Sigma\left\{\frac{1}{n^2}\sin^2 n\,m_0\,\pi\cos n(\theta+\theta_1)\right\}d\,\theta'\text{ henrys} \qquad (8)$$

$$= \frac{8\,N_1\,r\,l}{10^9\,\pi\,d}\,\Sigma\,\frac{1}{n^3}\sin^2 n\,m_0\,\pi\,\big[\sin n\,(\theta+\theta_1)\big]_{\theta'=-m_1\pi}^{\theta'=m_1\pi}\text{ henrys}$$

$$M_1 = \frac{16\,N_1\,r\,l}{10^9\,\pi\,d}\,\Sigma\left(\frac{1}{n^3}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi\cos n\,\theta\right)\text{ henrys} \qquad (9)$$

Next, if the loop of which M_1 is the mutual inductance is part of a winding having distribution density of winding $\frac{N_2}{2\,\pi}$ and subtending an angle $2\,m_2\,\pi$ its mutual inductance with the other winding will be

$$M_{12} = \frac{8\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\int_{-m_2\pi}^{m_2\pi}\Sigma\frac{1}{n^3}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi\cos n\,(\theta+\theta^1)\,d\,\theta'\text{ henry} \qquad (10)$$

$$= \frac{8\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\,\Sigma\,\frac{1}{n^4}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi\,\big[\sin n\,(\theta+\theta_1)\big]_{\theta'=-m_2\pi}^{\theta'=m_2\pi}\text{ henrys}$$

$$M_{12} = \frac{16\ N_1\ N_2\ r\ l}{10^9\ \pi^2\ d}\ \Sigma \left(\frac{1}{n^4} \sin^2 n\ m_0\ \pi \sin n\ m_1\ \pi \sin n\ m_2\ \pi \cos n\ \theta \right) \text{henrys} \quad \textbf{(11)}$$

This is the general expression for the mutual inductance between two groups of connected coils of like form on the same cylindrical core. It should be noted how much the harmonics have been reduced due to grouping.

When the coils are not of like design as in the case of a rotor and stator and the pitch of the coils is different in one from the other, sin $n\ m_0\ \pi$ will not appear twice in the equation but one of its values must be replaced by sin $n\ m_x\ \pi$ where $2\ m_x\ \pi$ is the pitch of the new coil. Equation (**11**) then becomes

$$M_{1a} = \frac{16\ N_1\ N_a\ r\ e}{10^9\ \pi^2\ d}\ \Sigma \left(\frac{1}{n^4} \sin n\ m_0\ \pi \sin n\ m_x\ \pi \sin n\ m_1\ \pi \sin n\ m_2\ \pi \cos n\ \theta \right) \text{henrys} \quad \textbf{(12)}$$

This formula is strictly correct when m_x is an integer and when θ is an integral multiple of the tooth pitch. It is true for all values of θ if either m_0 or m_x or both are unity.

By considering the axes of two similar groups of coils as coincident we obtain the value of $\Delta_1\ L_1$ which is part of the self inductance of the group, thus

$$\Delta_1\ L_1 = \frac{16\ N_1^2\ r\ e}{10^9\ \pi^2\ d} \Sigma \left(\frac{1}{n^4} \sin^2 n\ m_0\ \pi \sin^2 n\ m_1\ \pi \right) \quad \textbf{(13)}$$

The other factor that enters into the self inductance is the slot leakage inductance which depends upon the number of turns in a coil, the number of coils in a group and the width and depth of the slot and the length of the air gap. Since with the value of $\Delta_1\ L_1$ all the field which links the secondary winding has been included, only the portion of the slot leakage which does not link all the turns in the opposed secondary coil should be considered. No hard and fast rule can be made for determining this quantity since it depends upon the shape of the slots, there should be little trouble in making the calculation when the data is given. Denoting this quantity by $\Delta_2\ L_1$ we have

$$L_1 = \Delta_1\ L_1 + \Delta_2\ L_1 \quad \textbf{(14)}$$

Symmetrically Grouped Windings. The above formulae give the mutual impedance between groups of coils, each group of which may be unsymmetrical. Generally machines are designed so that, although the individual groups of coils due to fractional pitch may be unsymmetrical, the complete winding is symmetrical. When two coils are together in a slot this may be done by connecting one group of coils opposite the north pole in series with the corresponding group opposite the south pole; that is to say, the group displaced electrically by the angle π. If therefore we take equation (**11**) and consider the mutual induction as due to a group having axis at θ = zero and another having its axis at $\theta = \pi$ with a similarly arranged group of coils having its axis at θ, we find that (**11**) becomes

$$M_{12} = \frac{16\, N_1\, N_2\, r\, l}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin^2 n\, m_0\, \pi \sin n\, m_1\, \pi \right.$$

$$\left. \sin n\, m_2\, \pi\, (1 - \cos n\, \pi)^2 \cos n\, \theta \right\} \text{henrys} \qquad (\mathbf{15})$$

Similarly

$$M_{1a} = \frac{16\, N_1\, N_a\, r\, l}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin n\, m_1\, \pi \sin n\, m_x \right.$$

$$\left. \pi \sin n\, m_1\, \pi \sin n\, m_a\, \pi\, (1 - \cos n\, \pi)^2 \cos n\, \theta \right\} \text{henrys} \qquad (\mathbf{16})$$

Since $1 - \cos n\pi$ is zero for all even values of n it is evident that (**15**) and (**16**) contain no even harmonics, moreover the above formulae give the mutual induction between two similarly connected groups of windings, but if $(1 - \cos n\, \pi)$ is used only with the first power these formulas give the mutual impedance between one pair of such symmetrically grouped windings and another single group with axis inclined at an angle θ.

The value of self induction is

$$\Delta_1\, L_1 = \frac{16\, N_1^2\, r\, e}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin^2 n\, m_0\, \pi \sin^2 n\, m_1\, \pi \right.$$

$$\left. (1 - \cos n\, \pi)^2 \right\} \qquad (\mathbf{17})$$

$\Delta_2\, L_1$ is found in the same manner as before

$$L_1 = \Delta_1\, L_1 + \Delta_2\, L_1 \qquad (\mathbf{18})$$

It is obvious from (**15**) and (**16**) that the effect of dissymmetry is to introduce more or less double frequency into the wave form of generated e. m. f.

It will be seen from an examination of (**15**) and (**17**) that, for example, a winding of pitch $\frac{2\pi}{3}$ and subtending an angle $\frac{\pi}{3}$ when connected in a symmetrical group of two has the same field form and characteristics as a full pitch winding of the same number of turns subtending an angle $\frac{2\pi}{3}$.

There are many symmetrical forms of winding but all will be found to be covered by the formulas (**15**) and (**16**).

Unsymmetrical Windings. These may take many forms which may be classified:

(1) Dissymmetry of flux form due to even harmonics.

(2) Dissymmetry in axial position of polyphase groups.

(3) Dissymmetry in windings due to incorrect grouping of coils.

(4) Dissymmetry due to unsymmetrical magnetic characteristics of the iron.

Of these various forms of dissymmetry the most common is a combination of (**1**), (**2**) and (**3**). These forms of unsymmetrical windings may all be calculated by the formulas (**11**) to (**16**).

It is to be noted that the mutual inductance between a symmetrical and an unsymmetrical winding is harmonically symmetrical. Hence, if the field of a machine is harmonically symmetrical, the e. m. f. generated will be also harmonically symmetrical whatever may be the form of the windings.

The reciprocal nature of M is fully established by its form, for it is immaterial in obtaining (**16**) whether we start out with the winding whose pitch is m_x or with that whose pitch is m_0, the result will be the same. The effect of saturation will be to tend to alter the values of the coefficients of M but the general form will not vary appreciably. We shall now consider some standard windings of Generators and Motors.

Three-Phase Symmetrical Full Pitch. Here m_0, m_1 and m_2 are 0.5, 0.1666 and 0.1666 respectively. Using formula (**15**) all the even harmonics disappear and $(1 - \cos n\,\pi)^2$ is equal to 4 or zero.

$$M_{12} = \frac{16\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\left(\cos\theta + \frac{4}{81}\cos 3\,\theta + \frac{1}{625}\cos 5\,\theta + \frac{1}{2401}\cos 7\,\theta + \frac{4}{6561}\cos 9\,\theta + \ldots\right) \qquad (\mathbf{19})$$

Theoretical Symmetrical Three-Phase Winding. Here $m_0 = 0.5, m_1 = m_2 = 0.333.$ Using formula (**11**)

$$M_{12} = \frac{3}{4} \frac{16 N_1 N_2 r l}{10^9 \pi^2 d} \left(\cos \theta + \frac{1}{625} \cos 5 \theta + \frac{1}{2401} \cos 7 \theta + \frac{1}{14641} \cos 11 \theta + \ldots \right) \quad (20)$$

Here the third group of harmonics is entirely eliminated.

Three-Phase Symmetrical $\frac{2 \pi}{3}$ Pitch Winding. Here $m_0 = 0.333, m_1 = m_2 = 0.166.$ Using formula (**15**)

$$M = \frac{3}{4} \frac{16 N_1 N_2 r l}{10^9 \pi^2 a} \left(\cos \theta + \frac{1}{625} \cos 5 \theta + \frac{1}{2461} \cos 7 \theta + \frac{1}{14641} \cos 11 \theta + \ldots \right) \quad (21)$$

which gives the same result as (**20**).

Formulas for Salient Pole Machines

The formulas given in the preceding discussion are appropriate for distributed winding and non-salient poles. Where salient poles are used the field form due to the poles with a given winding will be arbitrary so that with the polar axis as reference we shall have

$$\mathcal{B} = \frac{2 \pi N_a I_a}{d} \Sigma \left(A_n \cos n \theta\right) \quad (22)$$

Where $\mathcal{B}$ is the induction through the armature or stator. When the poles are symmetrical $A_n \cos n \theta$ might be chosen at once for this condition and in this case we do not require coefficients of mutual induction between pole windings, since the value of $\mathcal{B}$ is obtained by considering the mutual reaction between pole windings to be such as will produce symmetry. We may however assume $\mathcal{B}$ to be perfectly general in form in which case the flux through a coil of pitch $2 m_0 \pi$ is

$$\varphi = \frac{4 \pi N_a I_a r l}{10 d} \Sigma \left(\frac{A_n}{n} \sin n m_0 \pi \cos n \theta\right) \quad (23)$$

We have therefore for the mutual induction between one pole and a group of coils at an angle θ and subtending an angle $2 m_1 \pi$

$$M_{a1} = \frac{4\ N_a\ N_1\ r\ l}{10^9\ d}\ \Sigma \left(\frac{A_n}{n^2} \sin n\ m_0\ \pi \sin n\ m_1\ \pi \cos n\ \theta \right) \quad (24)$$

and where there is symmetry due to grouping of windings, we have

$$M_{a1} = \frac{4\ N_a\ N_1\ r\ l}{10^9\ d}\ \Sigma \left\{ \frac{A_n}{n^2} \sin n\ m_0\ \pi \sin n\ m_1\ \pi \right.$$

$$\left. (1 - \cos n\ \pi)^2 \cos n\ \theta \right\} \quad (25)$$

where N_a is the number of turns for one pole and (**25**) applies to one pair of poles and the corresponding group of coils. When there are more than one pair of poles in series and the corresponding groups of winding are also in series, if it is desired to consider the mutual inductance of the complete winding, the result given above must be multiplied by the number of pairs of poles.

If in equation (**16**) we take

$$\left. \begin{aligned} \frac{N_a}{2\ \pi}\ \frac{1}{n} \sin n\ m_a\ \pi &= N_a \\ \text{and} \qquad \frac{1}{\pi\ n} &= B_n \end{aligned} \right\} \quad (26)$$

it becomes

$$M_{1a} = \frac{32\ N_1\ N_a\ r\ l}{10^9\ d}\ \Sigma \left\{ \frac{B_n}{n^2} \sin n\ m_x\ \pi \sin n\ m_0\ \pi \right.$$

$$\left. \sin n\ m_1\ \pi\ (1 - \cos n\ \pi)^2 \cos n\ \theta \right\} \quad (27)$$

which is the expression corresponding to (**25**) starting with the winding flux form. (**25**) and (**27**) must therefore be identical and we have

$$\frac{32\ N_1\ N_a\ r\ e}{10^9\ d}\ B_n \sin n\ m_x\ \pi - \frac{4\ N_a\ N_1\ r\ e}{10^9\ d}\ A_n$$

or

$$\mathcal{B}_n = \frac{A_u}{8 \sin n\ m_x\ \pi} \quad (28)$$

and

$$\mathcal{B}_1 = \frac{2\ \pi\ I_1}{10\ d}\ \Sigma\ (B_n \sin n\ m_0\ \pi \cos n\ \theta) \quad (29)$$

and is the induction wave form for a single turn of the winding.

The expression for the mutual inductance between windings of the same core for salient poles is obtained in terms of the pole flux wave form by substituting in the formulas $\frac{A_n}{8 \sin n\, m_x\, \pi}$ for $\frac{1}{n\, \pi}$. We have therefore the following formulas for salient poles.

General expression considering only one pole and one group of coils.

$$\mathcal{B}_a = \frac{2\, \pi\, N_a\, I_a}{10\, d}\, \Sigma\, (A_n \cos n\, \theta) \tag{a}$$

$$\mathcal{B}_1 = \frac{\pi\, I_1}{20\, d}\, \Sigma \left(A_n \frac{\sin n\, m_0\, \pi}{\sin n\, m_a\, \pi} \cos n\, \theta\right) \tag{b}$$

$$M_{a1} = \frac{4\, N_a\, N_1\, r\, l}{10^9\, d}\, \Sigma \left(\frac{A_n}{n^2} \sin n\, m_0\, \pi \sin n\, m_1\, \pi \cos n\, \theta\right) \tag{c}$$

$$M_{12} = \frac{2\, N_1\, N_2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left(\frac{A_n}{n^3} \frac{\sin n\, m_0\, \pi}{\sin n\, m_x\, \pi} \sin n\, m_0\, \pi \sin n\, m_1\, \pi \sin n\, m_2\, \pi \cos n\, \theta\right) \tag{d}$$

$$\Delta_1 L_a = \frac{4\, \pi\, N_a^2\, r\, l}{10^9\, d}\, \Sigma \left(\frac{A_n}{n} \sin n\, m_x\, \pi\right) \tag{e}$$

$$\Delta_1 L_1 = \frac{2\, N_1^2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left(\frac{A_n}{n} \frac{\sin^2 n\, m_0\, \pi \sin^2 n\, m_1\, \pi}{\sin n\, m_x\, \pi}\right) \tag{f}$$

General expressions considering only poles to be symmetrical. Considered on the basis of two poles, N_a being turns on one pole.

$$\mathcal{B}_a = \frac{2\, \pi\, N_a\, I_a}{10\, d}\, \Sigma\, \{A_n\, (1 - \cos n\, \pi) \cos n\, \theta\} \tag{a'}$$

$$\mathcal{B}_1 = \frac{\pi\, I_1}{20\, d}\, \Sigma \left\{A_n \frac{\sin n\, m_0\, \pi}{\sin n\, m_x\, \pi} (1 - \cos n\, \pi) \cos n\, \theta\right\} \tag{b'}$$

$$M_{a1} = \frac{4\, N_a'\, N_1\, r\, l}{10^9\, d}\, \Sigma \left\{\frac{A_n}{n^2} \sin n\, m_0\, \pi \sin n\, m_1\, \pi\, (1 - \cos n\, \pi) \cos n\, \theta\right\} \tag{c'}$$

$$M_{12} = \frac{2\,N_1\,N_2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left\{\frac{A_n}{n^3}\,\frac{\sin n\,m_0\,\pi}{\sin n\,m_x\,\pi}\,\sin n\,m_0\,\pi\,\sin n\,m_1\,\pi\right.$$

$$\left.\sin n\,m_2\,\pi\,\cos n\,\theta\right\}\text{henrys} \quad (\mathbf{d}')$$

$$\Delta_1\,L_a = \frac{4\,\pi\,N_a^2\,r\,l}{10^9\,d}\,\Sigma\left\{\frac{A_n}{n}\,\sin n\,m_x\,\pi\,(1 - \cos n\,\pi)\right\} \quad (\mathbf{e}')$$

$$\Delta_1\,L_1 = \frac{2\,N_1^2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left\{\frac{A_n}{n}\,\frac{\sin^2 n\,m_0\,\pi\,\sin^2 n\,m_1\,\pi}{\sin n\,m_x\,\pi}\right\} \quad (\mathbf{f}')$$

General expression with both polar and winding symmetry.

$$\mathfrak{B}_a = \frac{2\,\pi\,N_a\,I_a}{10\,d}\,\Sigma\,\{A_n\,(1 - \cos n\,\pi)\,\cos n\,\theta) \quad (\mathbf{a}'')$$

$$\mathfrak{B}_1 = \frac{\pi\,I_1}{20\,d}\,\Sigma\left\{A_n\,\frac{\sin n\,m_0\,\pi}{\sin n\,m_x\,\pi}\,(1 - \cos n\,\pi)\,\cos n\,\theta\right\} \quad (\mathbf{b}'')$$

$$M_{a1} = \frac{4\,N_a\,N_1\,r\,l}{10^9\,d}\,\Sigma\left\{\frac{A_n}{n^2}\,\sin n\,m_0\,\pi\,\sin n\,m_1\,\pi\,(1 - \cos n\,\pi)^2\right.$$

$$\left.\cos n\,\theta\right\} \quad (\mathbf{c}'')$$

$$M_{12} = \frac{2\,N_1\,N_2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left\{\frac{A_n}{n^3}\,\frac{\sin n\,m_0\,\pi}{\sin n\,m_x\,\pi}\,\sin n\,m_0\,\pi\,\sin n\,m_1\,\pi\right.$$

$$\left.\sin n\,m_2\,\pi\,(1 - \cos n\,\pi)^2\,\cos n\,\theta\right\} \quad (\mathbf{d}'')$$

$$\Delta_1\,L_a = \frac{4\,\pi\,N_a^2\,r\,l}{10^9\,d}\,\Sigma\left\{\frac{A_n}{n}\,\sin n\,m_x\,\pi\,(1 - \cos n\,\pi)^2\right\} \quad (\mathbf{e}'')$$

$$\Delta_1\,L_1 = \frac{2\,N_1^2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left\{\frac{A_n}{n}\,\frac{\sin^2 n\,m_0\,\pi\,\sin^2 n\,m_1\,\pi}{\sin n\,m_x\,\pi}\right.$$

$$\left.(1 - \cos n\,\pi)^2\right\} \quad (\mathbf{f}'')$$

In using any of the formulas given above for machines having more than two poles, it must be divided by the number of pairs of poles and likewise the expression for M or $\Delta_1\,L$ must be multiplied by the number of pairs of poles, which leaves the formula for these quantities unchanged.

Let us next consider the actual induction in the air gap with a distributed winding operating with three-phase currents. Let i_{m1} be the magnetizing current of the first phase i_{m2} and i_{m3} those of the other phases. The induction due to one group of coils of phase 1 is

$$\mathcal{B}_1 = \frac{8\, N_1\, i_{m1}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_1\, \pi \cos n\, \theta \right\} \tag{30}$$

and if the phase displacement of 2 and 3 from 1 be φ_{12} and φ_{13}

$$\mathcal{B}_2 = \frac{8\, N_2\, i_{m2}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_2\, \pi \cos (n\theta - \varphi_{12}) \right\} \tag{31}$$

$$\mathcal{B}_3 = \frac{8\, N_3\, i_{m3}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_2\, \pi \cos (n\, \theta - \varphi_{13}) \right\} \tag{32}$$

For symmetrically grouped coils the formulas become

$$\mathcal{B}_1 = \frac{8\, N_1\, i_{m1}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_1\, \pi\, (1 - \cos n\, \pi) \cos n\, \theta \right\} \tag{33}$$

$$\mathcal{B}_2 = \frac{8\, N_2\, i_{m2}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_2\, (1 - \cos n\, \pi) \cos m\, (\theta - \phi_{12}) \right\} \tag{34}$$

$$\mathcal{B}_3 = \frac{8\, N_3\, i_{m3}}{10\, \pi\, d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, m_3\, \pi\, (1 - \cos n\, \pi) \cos m\, (\theta - \varphi_{13}) \right\} \tag{35}$$

For a symmetrical three-phase motor with full pitch coils $m_0 = 0.5$, $m_1 = m_2 = m_3 = 0.166$ (**33**), (**39**) and (**35**) become of the four

$$\mathcal{B}_1 = \frac{8\, N_1\, i_{m1}}{10\, \pi\, d} \left\{ \cos \theta - \frac{2}{9} \cos 3\, \theta + \frac{1}{25} \cos 5\, \theta + \frac{1}{49} \cos 7\, \theta - \frac{2}{81} \cos 9\, \theta + \frac{1}{121} \cos 11\, \theta + \frac{1}{169} \cos 13\, \theta + \right\} \tag{36}$$

which is the field due to one group of coils alone. The wave is flattened by the third group of harmonics but all the other harmonics are peaking values. There is therefore a decided gain in such a wave form of flux since it permits of high fundamental flux density.

The maximum value of flux is approximately

$$B_{max} = 0.823 \cdot \frac{8\, N_1\, i_m}{10\, \pi\, d} \text{ gaus} \tag{37}$$

where d is given in centimeters.

$$B_{max} = \frac{1.67\, N_1\, i_m}{\pi\, d} \quad \text{maxwells per square inch,}$$

with d given in inches.

For the total winding the resultant induction will be the sum of B_1, B_2, and B. If we take the symmetrical winding with angles between planes of symmetry

$\varphi_{12} = \frac{2\,\pi}{3}$ and $\varphi_{13} = \frac{4\,\pi}{3}$, we have

$$\left.\begin{aligned} \cos n\,\theta &= \frac{e^{jn\theta}}{2} + \frac{e^{-jn\theta}}{2} \\ \cos n\left(\theta - \frac{2\,\pi}{3}\right) &= a^{-n}\,\frac{e^{jn\theta}}{2} + a^{n}\,\frac{e^{-jn\theta}}{2} \\ \cos n\left(\theta - \frac{4\,\pi}{3}\right) &= a^{n}\,\frac{e^{jn\theta}}{2} + a^{-n}\,\frac{e^{-jn\theta}}{2} \end{aligned}\right\} \tag{38}$$

If we multiply these three quantities successively by $\check{I}_{m1}$, $a^2\check{I}_{m1}$, $a\check{I}_{m1}$ and add, we have

$$\check{I}_{m1} \left\{ \frac{e^{jn\theta}}{2} \left(1 + a^{-(n-2)} + a^{(n+1)}\right) \left(+ \frac{e^{-jn\theta}}{2} \times \left(1 + a^{n+2} + a^{-(n-1)}\right) \right) \right\} \tag{39}$$

and giving n successive odd values from 1 up, we find for (**39**) the following values

$n = 1$ (**39**) becomes $\frac{3}{2}\, \check{I}_{m1}\, e^{-j\theta}$

$n = 3$ " " 0

$n = 5$ " " $\frac{3}{2} \check{I}_{m1} e^{j5\theta}$

$n = 7$ " " $\frac{3}{2} \check{I}_{m1} e^{-j7\theta}$

$n = 7$ " " 0

$n = 11$ " " $\frac{3}{2} \check{I}_{m1} e^{j11\theta}$

$n = n$ " " $2 \check{I}_{m1} \sin^2 \frac{2n\pi}{3} e^{-j\frac{2}{\sqrt{3}} \sin \frac{2n\pi}{3} n\theta}$

We may therefore express $\mathcal{B}$ by
$\mathcal{B}$ = real part of

$$\frac{16 N_1 \check{I}_{m1}}{10 \pi d} \Sigma \left\{ \frac{1}{n^2} \sin n\, m_0 \pi \sin n\, m_1 \pi (1 - \cos n\pi) \times \sin^2 \frac{2n\pi}{3} e^{-j\frac{2}{\sqrt{3}} \sin \frac{2n\pi}{3} n\theta} \right\} \tag{40}$$

It will be obvious that if we proceed around the cylinder in the negative direction of rotation at an angular speed w and $\check{I}_{m1}$ is equal to $I_{m1} e^{jwt}$, for $n = 1$ the value of B_1 will remain constant and real, hence B_1 must be a constant field rotating at angular velocity w in the negative direction. The value of B may be expressed in harmonic form, but in this form it does not illustrate the rotating field theory so aptly. The harmonic form is given below and is simpler in appearance than (**40**).

$$\mathcal{B} = \frac{16 N_1 i_{m1}}{10 \pi d} \Sigma \left(\frac{1}{n^2} \sin n\, m_0 \pi \sin n\, m_1 \pi (1 - \cos n\pi) \sin^2 \frac{2n\pi}{3} \cos n\theta \right) \tag{41}$$

For a symmetrical three-phase motor with full pitch coil ($m_0 = 0.5\, m_1 = 0.166$) $\mathcal{B}$ becomes

$$\mathcal{B} = \frac{12 N_1 i_{m1}}{10 \pi d} \Sigma \left\{ \cos\theta + \frac{1}{25} \cos 5\theta + \frac{1}{49} \cos 7\theta + \frac{1}{121} \cos 11\theta + \frac{1}{169} \cos 13\theta + \ldots \right\} \tag{42}$$

This gives for the maximum induction approximately

$$\mathfrak{B}_{max} = \frac{1.075 \times 12\, N_1\, i_{m1}}{10\, \pi\, d} = \frac{1.29\, N_1\, i_m}{\pi\, d} \text{ gauss} \tag{43}$$

where d is measured in centimeters.

$$\mathfrak{B}_{max} = \frac{3.28 \times N_1\, i_{m1}}{\pi\, d} \text{ maxwell per square inch} \tag{44}$$

where d is measured in inches and N is the total number of turns per pair of poles.