SULL' EQUILIBRIO DEI CORPI ELASTICI ISOTROPI ; NOTA DEL DOTT.

GIUSEPPE LAURICELLA [1]).

Ciascuno degli integrali che compariscono nel secondo membro è uniformemente proprio; ossia tolto dal campo S_1 un campo sufficientemente piccolo s_1 contenente nel suo interno il punto di infinito, l' integrale corrispondente esteso ad s_1 si mantiene in valore assoluto inferiore ad una quantità positiva ε piccola a piacere, anche coll'indefinito impiccolire di Δx_1. Infatti per i primi tre basta ripetere i ragionamenti del Morera; per il quarto integrale basta osservare che $\frac{(x-x_1)^2}{r^2} = \cos^2\theta \leq 1$; per il quinto che $\left|\frac{x-x_1}{r}\right| = |\cos\theta| \leq 1$, $\left|\frac{\Delta x_1}{r}\right| \leq 2$; per il sesto che $\frac{\Delta x_1^2}{r^2} \leq 4$; per il settimo che

$$\frac{1}{r^4} + \frac{1}{r^3 r_1} + \frac{1}{r^2 r_1^2} + \frac{1}{r r_1^3} + \frac{1}{r_1^4} \leq \frac{5}{r_1^4} , \quad \frac{(x_1+\Delta x_1-x)^4}{r_1^3(r+r_1)} \leq 1 ;$$

per l' ultimo che

$$\frac{1}{r^4} + \frac{1}{r^3 r_1} + \frac{1}{r^2 r_1^2} + \frac{1}{r r_1^3} + \frac{1}{r_1^4} \leq \frac{5}{r_1^4} , \quad \left|\frac{(x_1+\Delta x_1-x)^3 \Delta x_1}{r_1^3(r_1+r)}\right| \leq 1.$$

Se invece $r < r_1$, avremo $\frac{1}{r} > \frac{1}{r_1}$, ed inoltre :

$$\Delta \frac{\partial u_1}{\partial x_1} = \left(\frac{x-x_1-\Delta x_1}{r_1^3} - \frac{x-x_1-\Delta x_1}{r^3} - \frac{\Delta x_1}{r^3}\right)\left(1 + \frac{3\alpha}{2}\right) - \frac{3\alpha}{2}\left(\frac{(x-x_1)^3}{r_1^5} \right.$$
$$\left. - \frac{(x-x_1)^3}{r^5} - 3\,\frac{(x-x_1)^2 \Delta x_1}{r_1^5} + 3\,\frac{(x-x_1)\Delta x_1^2}{r_1^5} - \frac{\Delta x_1^3}{r_1^5}\right)$$
$$= \left\{(x-x_1-\Delta x_1)\left(\frac{1}{r_1^3} - \frac{1}{r^3}\right) - \frac{\Delta x_1}{r^3}\right\}\left(1 + \frac{3\alpha}{2}\right) - \frac{3\alpha}{2}\left[(x-x_1)^3\left(\frac{1}{r_1^5} - \frac{1}{r^5}\right)\right.$$
$$\left. - 3\,\frac{(x-x_1)^2\Delta x_1}{r_1^5} + 3\,\frac{(x-x_1)\Delta x_1^2}{r_1^5} - \frac{\Delta x_1^3}{r_1^5}\right]$$
$$= \left\{(x-x_1-\Delta x_1)\left(\frac{1}{r_1^3} - \frac{1}{r^3}\right) - \frac{\Delta x_1}{r^3}\right\}\left(1 + \frac{3\alpha}{2}\right) - \frac{3\alpha}{2}\left[\frac{(x-x_1)^3}{r} \cdot \frac{\Delta r}{r_1}\left(\frac{1}{r_1^4}\right.\right.$$
$$\left.\left. + \frac{1}{r_1^3 r} + \frac{1}{r_1^2 r^2} + \frac{1}{r_1 r^3} + \frac{1}{r^4}\right) - 3\,\frac{(x-x_1)^2\,\Delta x_1}{r_1^5} + 3\,\frac{(x-x_1)\,\Delta x_1^2}{r_1^5} - \frac{\Delta x_1^3}{r_1^5}\right] ;$$

1) *Continuazione e fine.* Vedi p. 141.

per cui:

$$\frac{\Delta\frac{\partial u_1}{\partial x_1}}{\Delta x_1}=\left\{-\frac{1}{r^3}+\frac{2}{r}\frac{(x_1+\Delta x_1-x)^2}{r_1(r+r_1)}\left(\frac{1}{r_1^2}+\frac{1}{r_1 r}+\frac{1}{r^2}\right)-\frac{\Delta x_1}{r}\cdot\frac{x_1+\Delta x_1-x}{r_1(r+r_1)}\left(\frac{1}{r_1^2}\right.\right.$$

$$\left.\left.+\frac{1}{r_1 r}+\frac{1}{r^2}\right)\right\}\left(1+\frac{3\alpha}{2}\right)-\frac{3\alpha}{2}\left\{-3\frac{(x-x_1)^2}{r_1^5}+3\frac{(x-x_1)\Delta x_1}{r_1^5}-\frac{\Delta x_1^2}{r_1^5}\right.$$

$$+2\frac{(x_1-x)^3}{r(r+r_1)}\cdot\frac{(x_1+\Delta x_1-x)}{r_1}\left(\frac{1}{r_1^4}+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\right)$$

$$\left.-\frac{(x_1+\Delta x_1-x)\Delta x_1}{r_1(r+r_1)}\cdot\frac{(x_1-x)^3}{r}\left(\frac{1}{r_1^4}+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\right)\right\}.$$

Avremo dunque:

$$(9)\quad \int_{S_2}(\rho X-\rho_0 X_0)\frac{\Delta\frac{\partial u_1}{\partial x_1}}{\Delta x_1}dS_2=\left[-\int_{S_2}\frac{\rho X-\rho_0 X_0}{r}\cdot\frac{dS_2}{r_2}\right.$$

$$+2\int_{S_2}\frac{\rho X-\rho_0 X_0}{r}\cdot\frac{(x_1+\Delta x_1-x)^2}{r_1(r+r_1)}\left(\frac{1}{r_1^2}+\frac{1}{r_1 r}+\frac{1}{r^2}\right)dS_2$$

$$\left.-\int_{S_2}\frac{\rho X-\rho_0 X_0}{r}\frac{(x_1+\Delta x_1-x)\Delta x_1}{r_1(r+r_1)}\left(\frac{1}{r_1^2}+\frac{1}{r_1 r}+\frac{1}{r^2}\right)dS_2\right]\left(1+\frac{3\alpha}{2}\right)$$

$$-\left[-3\int_{S_2}\frac{\rho X-\rho_0 X_0}{r_1}\cdot\frac{(x-x_1)^2}{r_1^2}\cdot\frac{dS_2}{r_1^2}\right.$$

$$+3\int_{S_2}\frac{\rho X-\rho_0 X_0}{r_1}\cdot\frac{(x-x_1)}{r_1}\cdot\frac{\Delta x_1}{r_1}\cdot\frac{dS_2}{r_1^2}-\int_{S_2}\frac{\rho X-\rho_0 X_0}{r_1}\cdot\frac{\Delta x_1^2}{r_1^4}dS_2$$

$$+2\int_{S_2}\frac{\rho X-\rho_0 X_0}{r}\cdot\frac{(x_1-x)^3}{r+r_1}\cdot\frac{(x_1+\Delta x_1-x)}{r_1}\left(\frac{1}{r_1^4}\right.$$

$$\left.+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\right)dS_2-\int_{S_2}\frac{\rho X-\rho_0 X_0}{r}\cdot\frac{(x_1+\Delta x_1-x)\Delta x_1}{r_1(r+r_1)}\cdot\frac{(x_1-x)^3}{r}\left(\frac{1}{r_1^4}\right.$$

$$\left.\left.+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\right)dS_2\right]\frac{3\alpha}{2}.$$

Gli integrali del secondo membro della precedente uguaglianza sono tutti uniformemente proprî; infatti i primi tre sono simili ai primi tre della (8); gli altri risultano proprî dall'osservare che si ha:

$$\left|\frac{\rho X-\rho_0 X_0}{r_1}\right|\frac{(x-x_1)^2}{r_1^2}\leq\left|\frac{\rho X-\rho_0 X_0}{r}\right|\frac{(x-x_1)^2}{r^2}=\left|\frac{\rho X-\rho_0 X_0}{r}\right|\cos^2\theta\leq\left|\frac{\rho X-\rho_0 X_0}{r}\right|;$$

$$\left|\frac{\rho X-\rho_0 X_0}{r_1}\cdot\frac{(x-x_1)}{r_1}\right|\leq\left|\frac{\rho X-\rho_0 X_0}{r}\cdot\frac{(x-x_1)}{r}\right|\leq\left|\frac{\rho X-\rho_0 X_0}{r}\right|,\quad\left|\frac{\Delta x_1}{r_1}\right|<2;$$

$$\left|\frac{\rho X-\rho_0 X_0}{r_1}\right|\frac{\Delta x_1^2}{r_1^2}\leq 4\left|\frac{\rho X-\rho_0 X_0}{r}\right|;$$

$$\frac{1}{r_1^4}+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\leq\frac{5}{r^4},\quad\left|\frac{(x_1-x)^3}{r^2(r+r_1)}\cdot\frac{(x_1+\Delta x_1-x)}{r_1}\right|\leq 1;$$

$$\frac{1}{r_1^4}+\frac{1}{r_1^3 r}+\frac{1}{r_1^2 r^2}+\frac{1}{r_1 r^3}+\frac{1}{r^4}\leq\frac{5}{r^4},\quad\left|\frac{\Delta x(x_1+\Delta x_1-x)}{r_1(r+r_1)}\cdot\frac{(x_1-x)^3}{r^3}\right|\leq 1$$

Se indichiamo con S'_1, S'_2 ciò che divengono i due campi S_1, S_2, quando il piano normale a Δx_1 è arrivato in (x_1, y_1, z_1), avremo:

$$S=S_1+S_2=S'_1+S'_2,$$

$$\lim_{\Delta x_1=0}\int_S(\rho X-\rho_0 X_0)\frac{\Delta\frac{\partial u_1}{\partial x_1}}{\Delta x_1}dS=\lim_{\Delta x_1=0}\int_{S_1}(\rho X-\rho_0 X_0)\frac{\Delta\frac{\partial u_1}{\partial x_1}}{\Delta x_1}dS_1$$

$$+\lim_{\Delta x_1=0}\int_{S_2}(\rho X-\rho_0 X_0)\frac{\Delta\frac{\partial u_1}{\partial x_1}}{\Delta x_1}dS_2.$$

Gli integrali che compariscono nei secondi membri delle (8) e delle (9), come abbiamo veduto, sono uniformemente proprî; per cui, supposto Δx_1 minore di un certo limite, possiamo dal campo S togliere un campo $s=s_1+s_2$ contenente nel suo interno i punti (x_1, y_1, z_1), $(x_1+\Delta x_1, y_1, z_1)$, in modo che gli integrali estesi ai nuovi campi S_1-s_1, S_2-s_2 differiscano rispettivamente dagli integrali (8), (9) di tanto poco quanto si vuole. Posto ciò, gli integrali che contengono sotto il segno di integrazione il fattore Δx_1, estesi rispettivamente ai campi $S_1 - s_1$, $S_2 - s_2$ possono rendersi minori di qualunque grandezza assegnabile per Δx_1 sufficientemente piccolo, al limite quindi per $\Delta x_1 = 0$ gli integrali analoghi estesi ai campi S_1, S_2 vanno a zero. Abbiamo dunque:

$$\lim_{\Delta x_1 = 0} \int_{S_1} (\rho X - \rho_0 X_0) \frac{\Delta \frac{\partial u_1}{\partial x_1}}{\Delta x_1} dS_1 = \left[\int_{S'_1} (\rho X - \rho_0 X_0) \left\{ - \frac{1}{r^3} + 3 \frac{(x - x_1)^2}{r^5} \right\} dS'_1 \right] \left(1 + \frac{3\alpha}{2} \right) - \left[\int_{S'_1} (\rho X - \rho_0 X_0) \left\{ - 3 \frac{(x - x_1)^2}{r^5} + 5 \frac{(x - x_1)^4}{r^7} \right\} d S'_1 \right] \frac{3\alpha}{2} ,$$

$$\lim_{\Delta x_1 = 0} \int_{S_2} (\rho X - \rho_0 X_0) \frac{\Delta \frac{\partial u_1}{\partial x_1}}{\Delta x_1} dS_2 = \left[\int_{S'_2} (\rho X - \rho_0 X_0) \left\{ - \frac{1}{r^3} + 3 \frac{(x - x_1)^2}{r^5} \right\} d S'_2 \right] \left(1 + \frac{3\alpha}{2} \right) - \left[\int_{S'_2} (\rho X - \rho_0 X_0) \left\{ - 3 \frac{(x - x_1)^2}{r^5} + 5 \frac{(x - x_1)^4}{r^7} \right\} d S'_2 \right] \frac{3\alpha}{2} ;$$

e finalmente:

$$\lim_{\Delta x_1 = 0} \int_S (\rho X - \rho_0 X_0) \frac{\Delta \frac{\partial u_1}{\partial x_1}}{\Delta x_1} dS = \int_S (\rho X - \rho_0 X_0) \left\{ - \frac{1}{r^3} + 3 \frac{(x - x_1)^2}{r^5} + \frac{3\alpha}{2} \left(- \frac{1}{r^3} + 6 \frac{(x - x_1)^2}{r^5} - 5 \frac{(x - x_1)^4}{r^7} \right) \right\} dS = \int_S (\rho X - \rho_0 X_0) \frac{\partial^2 u_1}{\partial x_1^2} dS.$$

Analogamente si potrebbe dimostrare che

$$\lim_{\Delta x_1 = 0} \int_S (\rho Y - \rho_0 Y_0) \frac{\Delta \frac{\partial v_1}{\partial x_1}}{\Delta x_1} dS = \int_S (\rho Y - \rho_0 Y_0) \frac{\partial^2 v_1}{\partial x_1^2} dS ,$$

$$\lim_{\Delta x_1 = 0} \int_S (\rho Z - \rho_0 Z_0) \frac{\Delta \frac{\partial w_1}{\partial x_1}}{\Delta x_1} dS = \int_S (\rho Z - \rho_0 Z_0) \frac{\partial^2 w_1}{\partial x_1^2} dS ;$$

sicchè possiamo scrivere:

$$\frac{\partial^2 M}{\partial x_1^2} = -\Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_1}{\partial x} \frac{\partial x}{\partial n} d\sigma + \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial^2 u_1}{\partial x_1^2} dS \,,$$

e così:

$$\frac{\partial^2 M}{\partial y_1^2} = -\Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_1}{\partial y} \frac{\partial y}{\partial n} d\sigma + \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial^2 u_1}{\partial y_1^2} dS \,,$$

$$\frac{\partial^2 M}{\partial z_1^2} = -\Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_1}{\partial z} \frac{\partial z}{\partial n} d\sigma + \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial^2 u_1}{\partial z_1^2} dS \,,$$

$$\frac{\partial^2 N}{\partial x_1 \partial y_1} = -\Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_2}{\partial x} \frac{\partial y}{\partial n} d\sigma + \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial^2 u_2}{\partial x_1 \partial y_1} dS \,,$$

$$\frac{\partial^2 P}{\partial x_1 \partial z_1} = -\Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_3}{\partial x} \frac{\partial z}{\partial n} d\sigma + \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial^2 u_3}{\partial x_1 \partial z_1} dS \,.$$

Posto dunque:

$$\frac{\partial M}{\partial x_1} + \frac{\partial N}{\partial y_1} + \frac{\partial P}{\partial z_1} = \theta \,,$$

avremo:

$$\begin{aligned} L\Delta^2 M + (L+K) \frac{\partial \theta}{\partial x_1} = &- L \Sigma \rho_0 X_0 \int_\sigma \frac{\partial u_1}{\partial n} d\sigma - (L+K) \Sigma \rho_0 X_0 \int_\sigma \Big(\frac{\partial u_1}{\partial x} \frac{\partial x}{\partial n} \\ &+ \frac{\partial u_2}{\partial x} \frac{\partial y}{\partial n} + \frac{\partial u_3}{\partial x} \frac{\partial z}{\partial n} \Big) d\sigma + L \int_S \Sigma (\rho X - \rho_0 X_0) \Delta^2 u_1 \, dS \\ &+ (L+K) \int_S \Sigma (\rho X - \rho_0 X_0) \frac{\partial}{\partial x_1} \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_3}{\partial z_1} \Big) dS \,. \end{aligned}$$

Ora:

$$L\Delta^2 u_1 + (L+K) \frac{\partial}{\partial x_1} \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_3}{\partial z_1} \Big) = 0 \,,$$

quindi:

$$L\Delta^2 M + (L+K)\frac{\partial\theta}{\partial x_1} = -L\,\Sigma\rho_0 X_0 \int_\sigma \frac{\partial u_1}{\partial n}\,d\sigma - (L+K)\,\Sigma\,\rho_0 X_0 \int_\sigma \left(\frac{\partial u_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial u_3}{\partial x}\frac{\partial z}{\partial n}\right) d\sigma \,.$$

Si ha intanto:

$$L\rho_0 X_0 \frac{\partial u_1}{\partial n} = L\rho_0 X_0 \frac{\partial \frac{1}{r}}{\partial n} + L\rho_0 X_0 \frac{\alpha}{2}\frac{\partial}{\partial n}\frac{\partial^2 r}{\partial x^2} = L\rho_0 X_0 \frac{\partial \frac{1}{r}}{\partial n}$$

$$+ L\rho_0 X_0 \frac{\alpha}{2}\left(\frac{\partial^3 r}{\partial x^3}\frac{\partial x}{\partial n} + \frac{\partial^3 r}{\partial x^2\partial y}\frac{\partial y}{\partial n} + \frac{\partial^3 r}{\partial x^2 \partial z}\frac{\partial z}{\partial n}\right) = L\,\rho_0 X_0 \frac{\partial \frac{1}{r}}{\partial n}$$

$$+ L\rho_0 X_0 \left(\frac{\partial u_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial u_3}{\partial x}\frac{\partial z}{\partial n}\right) - L\rho_0 X_0 \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial x}{\partial n}\,,$$

$$L\rho_0 Y_0 \frac{\partial v_1}{\partial n} = L\rho_0 Y_0 \left(\frac{\partial v_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial v_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial v_3}{\partial x}\frac{\partial z}{\partial n}\right) - L\rho_0 Y_0 \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial y}{\partial n}\,,$$

$$L\rho_0 Z_0 \frac{\partial w_1}{\partial n} = L\rho_0 Z_0 \left(\frac{\partial w_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial w_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial w_3}{\partial x}\frac{\partial z}{\partial n}\right) - L\rho_0 Z_0 \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial z}{\partial n}\,,$$

per cui:

$$L\Delta^2 M + (L+K)\frac{\partial\theta}{\partial x_1} = -L\rho_0 X_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial n}\,d\sigma$$

$$-(2L+K)\rho_0 X_0 \int_\sigma \left(\frac{\partial u_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial u_3}{\partial x}\frac{\partial z}{\partial n}\right) d\sigma + L\rho_0 X_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial x}{\partial n}\,d\sigma$$

$$-(2L+K)\rho_0 Y_0 \int_\sigma \left(\frac{\partial v_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial v_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial v_3}{\partial x}\frac{\partial z}{\partial n}\right) d\sigma + L\rho_0 Y_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial y}{\partial n}\,d\sigma$$

$$-(2L+K)\,\rho_0 Z_0 \int_\sigma \left(\frac{\partial w_1}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial w_2}{\partial x}\frac{\partial y}{\partial n} + \frac{\partial w_3}{\partial x}\frac{\partial z}{\partial n}\right) d\sigma + L\rho_0 Z_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial z}{\partial n}\,d\sigma\,.$$

Osserviamo che la formula (5) l'abbiamo stabilita indipendentemente dai valori di $\rho_0 X_0$, $\rho_0 Y_0$, $\rho_0 Z_0$, sicchè possiamo scrivere:

$$\frac{\partial}{\partial x}\int_S u_1\, dS = \int_S \frac{\partial u_1}{\partial x}\, dS = -\int_\sigma u_1 \frac{\partial x}{\partial n}\, d\sigma \,.$$

e così:

$$\frac{\partial}{\partial y}\int_S u_1\, dS = \int_S \frac{\partial u_1}{\partial y}\, dS = -\int_\sigma u_1 \frac{\partial y}{\partial n}\, d\sigma \,.$$

Derivando la prima delle precedenti formule rispetto ad y, la seconda rispetto ad x, otteniamo:

$$\frac{\partial^2}{\partial x \partial y}\int_S u_1\, dS = -\int_\sigma \frac{\partial u_1}{\partial y}\frac{\partial x}{\partial n}\, d\sigma = -\int_\sigma \frac{\partial u_1}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \;;$$

dunque

$$\int_\sigma \frac{\partial u_1}{\partial y}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial u_1}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \,,$$

ed analogamente:

$$\int_\sigma \frac{\partial u_2}{\partial y}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial u_2}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \,,\; \int_\sigma \frac{\partial u_3}{\partial z}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial u_3}{\partial x}\frac{\partial z}{\partial n}\, d\sigma \,,$$

$$\int_\sigma \frac{\partial v_2}{\partial y}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial v_2}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \,,\; \int_\sigma \frac{\partial v_3}{\partial z}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial v_3}{\partial x}\frac{\partial z}{\partial u}\, d\sigma \,,$$

$$\int_\sigma \frac{\partial w_2}{\partial y}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial w_2}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \,,\; \int_\sigma \frac{\partial w_3}{\partial z}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial w_3}{\partial x}\frac{\partial z}{\partial n}\, d\sigma$$

$$\int_\sigma \frac{\partial \frac{1}{r}}{dy}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial y}{\partial n}\, d\sigma \,,\; \int_\sigma \frac{\partial \frac{1}{r}}{\partial z}\frac{\partial x}{\partial n}\, d\sigma = \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial z}{\partial n}\, d\sigma \,.$$

Si ha quindi:

$$L\Delta^2 M + (L+K)\frac{\partial\theta}{\partial x_1} = -L\rho_0 X_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial n} d\sigma$$

$$-(2L+K)\rho_0 X_0 \int_\sigma \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right)\frac{\partial x}{\partial n} d\sigma + L\rho_0 X_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial x}{\partial n} d\sigma$$

$$-(2L+K)\rho_0 Y_0 \int_\sigma \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\frac{\partial x}{\partial n} d\sigma + L\rho_0 Y_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial y}{\partial n} d\sigma$$

$$-(2L+K)\rho_0 Z_0 \int_\sigma \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}\right)\frac{\partial x}{\partial n} d\sigma + L\rho_0 Z_0 \int_\sigma \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial z}{\partial n} d\sigma \; ;$$

ma:

$$(2L+K)\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = (2L+K)\left(\frac{\partial \frac{1}{r}}{\partial x} + \frac{\alpha}{2}\frac{\partial}{\partial x}\Delta^2 r\right)$$

$$= (2L+K)(1+\alpha)\frac{\partial \frac{1}{r}}{\partial x} = L\frac{\partial \frac{1}{r}}{\partial x} \, ,$$

$$(2L+K)\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) = L\frac{\partial \frac{1}{r}}{\partial y} \, ,$$

$$(2L+K)\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}\right) = L\frac{\partial \frac{1}{r}}{\partial z} \, ,$$

dunque:

$$L\Delta^2 M + (L+K)\frac{\partial\theta}{\partial x_1} = -4\pi L\rho_0 X_0 + L\rho_0 X_0 \int_\sigma \left(\frac{\partial \frac{1}{r}}{\partial x}\frac{\partial x}{\partial n} - \frac{\partial \frac{1}{r}}{\partial x}\frac{\partial x}{\partial n}\right) d\sigma$$

$$+L\rho_0 Y_0 \int_\sigma \left(\frac{\partial \frac{1}{r}}{\partial x}\frac{\partial y}{\partial n} - \frac{\partial \frac{1}{r}}{\partial y}\frac{\partial x}{\partial n}\right) d\sigma + L\rho_0 Z_0 \int_\sigma \left(\frac{\partial \frac{1}{r}}{\partial x}\frac{\partial z}{\partial n} - \frac{\partial \frac{1}{r}}{\partial z}\frac{\partial x}{\partial n}\right) d\sigma$$

$$= -4\pi L\rho_0 X_0 .$$

Similmente:

$$L\Delta^2 N + (L + K)\frac{\partial\theta}{\partial y_1} = -4\pi L\rho_0 Y_0 ,$$

$$L\Delta^2 P + (L + K)\frac{\partial\theta}{\partial z_1} = -4\pi L\rho_0 Z_0 ,$$

Osservazione. — Poichè le espressioni:

$$\int_\sigma \Sigma X_\sigma u_1 d\sigma - \int_\sigma \Sigma X_\sigma^{(1)} u\, d\sigma ,$$

$$\int_\sigma \Sigma X_\sigma u_2 d\sigma - \int_\sigma \Sigma X_\sigma^{(2)} u\, d\sigma ,$$

$$\int_\sigma \Sigma X_\sigma u_3 d\sigma - \int_\sigma \Sigma X_\sigma^{(3)} u\, d\sigma$$

soddisfano alle equazioni dell' equilibrio scritte nel § 1 per il caso di $\rho X = \rho Y = \rho Z = 0$ (Vedi *Somigliana*, l. c., § 1), risulta che le formule (1) del Somigliana valgono anche quando le funzioni ρX, ρY, ρZ soddisfano alle sole condizioni che abbiamo poste nel § 3.