ORIGINAL COMMUNICATIONS.

NOTE ON AIR-GAP AND INTERPOLAR INDUCTION.

By F. W. CARTER, M.A., Associate.

The publication of Messrs. Hawkins and Wightman's recent dissertation on the "Air-gap Induction in Continuous Current Dynamos"[1] induces me to present to the Institution the following note on the subject of the fringe of lines near the edge of a pole-piece. It need hardly be pointed out in this place that the exact solution of the magnetic problem presented by a dynamo is impossible without an almost inconceivable addition to our present mathematical knowledge. Hence we are driven to making various assumptions about the shape of the lines of force in order to obtain a more or less approximate estimate of the strength of the field at the armature surface. Such assumptions often lead to sufficiently accurate results over limited regions, but are liable to break down where a solution is most wanted. Thus it is practically correct to assume that the lines of force pass straight across the air-gap, and that they form arcs of circles outside the air-gap—*so long as we are not near the edge of the pole-piece*, but the solution fails in the immediate neighbourhood of that edge. Messrs. Hawkins and Wightman's process of rounding the corner of the curve at discretion[2] can hardly be regarded as satisfactory.

In the following I propose to give the exact solution of some comparatively simple, though ideal, problems concerning the field of force near the edge of a pole-piece, in order to provide a criterion whereby to judge of the merits of the various approximate formulæ used in this connection, to indicate the order of accuracy that may be expected from the use of these formulæ, and to furnish a plausible means of correcting the results which they lead to.

The first problem to be discussed is the two-dimensional one of the lines of force from the pole-piece to the armature shown in Fig. 1, when the pole-piece is supposed to be at one magnetic potential, and the armature at another.

[1] See this Volume, p. 436, seq. [2] See this Volume, p. 440.

As the theory of conjugate functions is somewhat out of the province of the electrical engineer, I shall content myself with giving only so much of the mathematics as will enable those interested in such subjects to verify my results.

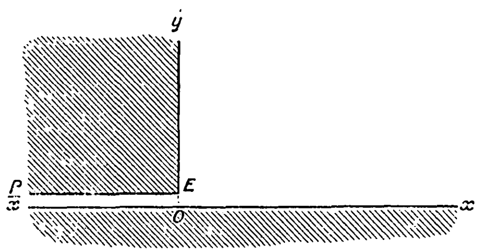

FIG. 1.—The Plane of $x\,y$.

Briefly then, we seek a potential function, ϕ, which reduces to the constant value ϕ_0 at the pole-piece, and to zero at the armature. Since the flux depends only on difference of potential, there is no loss of generality in assuming the potential at the armature surface to be zero.

Now the transformation—

$$z = \frac{2g}{\pi}\left\{\frac{\sqrt{a^2-\zeta^2}}{a} - \log\left(\frac{a+\sqrt{a^2-\zeta^2}}{-\zeta}\right) + i\frac{\pi}{2}\right\},$$

(where $z = x + i\,y$, $\zeta = \xi + i\,\eta$, $i = \sqrt{-1}$ and g is the length of the air-gap,)

converts the quadrant $\eta\, o\, \bar{\xi}$ shown in Fig. 2, into the region we require, bounded by the pole-piece and armature of Fig. 1,—making O A into the pole face P E, A $\bar{\xi}$ into the side

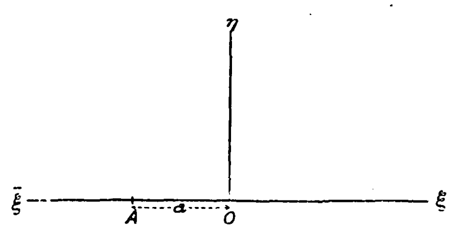

FIG. 2.—The Plane of $\xi\,\eta$.

E y of the pole, and $o\,\eta$ into the armature face $\bar{x}\,x$. For on O A, ζ is real, negative, and numerically less than a; thus Z = a negative real part + $i\,g$, *i.e.*, $y = g$ and x is negative—corresponding to P E; on A $\bar{\xi}$, z is a pure imaginary, between

$i\,g$ and ∞, *i.e.*, $x = o$ and $y > g$,—corresponding to Ey; on $o\,\eta$, ζ is a pure imaginary $= i\,\eta$, and z is real, varying from $-\infty$ to ∞. Now if $\phi + i\,\psi$ be such a function of ζ, that ϕ is equal to ϕ_0 along $o\,\xi$, and to zero along $o\,\eta$, then this function transformed into $x\,y$ co-ordinates by means of the above relation, will give us the sought-for equipotential surfaces (ϕ = constant) and lines of force (ψ = constant).

The function having the above properties is given by

$$\zeta = e^{\frac{\pi}{2} i \left(\frac{\phi + \phi_0 + i\psi}{\phi_0}\right)},$$

where e is the base of the natural logarithms, $(=2{\cdot}718\ .\ .)$.

Thus putting $a = e^{\frac{\pi}{2}\frac{\psi_0}{\phi_0}}$, we get

$$z = \frac{2g}{\pi}\left\{\sqrt{1 - e^{\pi i\left(\frac{\phi + \phi_0 + i\psi + \psi_0}{\phi_0}\right)}} - \log\left[e^{-\frac{\pi}{2} i \left(\frac{\phi - \phi_0 + i\psi + \psi_0}{\phi_0}\right)} + \sqrt{e^{-\pi i\left(\frac{\phi - \phi_0 + i\psi + \psi_0}{\phi_0}\right)} - 1}\,\right] + i\frac{\pi}{2}\right\}$$

We cannot express ϕ and ψ explicitly in terms of x and y, but we can nevertheless trace the equipotential lines, ϕ = constant, and lines of force, ψ = constant, from the above. We are, however, only concerned with the strength of the field at the armature surface, that is, with the value of $-\frac{\delta\phi}{\delta y}$, or of $\frac{\delta\psi}{\delta x}$, when y is zero.

Putting $\phi = o$, we get

$$x = \frac{2g}{\pi}\left\{\sqrt{1 + e^{-\pi\frac{\psi + \psi_0}{\phi_0}}} - \log\left[e^{\frac{\pi}{2} i}\left(e^{\frac{\pi}{2}\frac{\psi + \psi_0}{\phi_0}} + \sqrt{e^{\pi\frac{\psi + \psi_0}{\phi_0}} + 1}\right)\right] + i\frac{\pi}{2}\right\}$$

$$= \frac{2g}{\pi}\left\{\sqrt{1 + e^{-2X}} - \log\left(e^{X} + \sqrt{1 + e^{2X}}\right)\right\}\ .\ \text{i}$$

where $X = \frac{\pi}{2} \frac{\psi + \psi_0}{\phi_0}$

$$\therefore \frac{dx}{d\psi} = \frac{\pi}{2\phi_0} \frac{dx}{dX}$$

$$= \frac{g}{\phi_0}\left\{ \frac{-e^{-2X}}{\sqrt{1+e^{-2X}}} - \frac{e^{X}}{\sqrt{1+e^{2X}}} \right\}$$

$$= -\frac{g}{\phi_0} \frac{e^{X}+e^{-X}}{\sqrt{1+e^{2X}}}$$

$$\therefore \text{ the force } F = \frac{\phi_0}{g} \cdot \frac{\sqrt{1+e^{2X}}}{e^{X}+e^{-X}}$$

$$= F_0 \frac{1}{\sqrt{1+e^{-2X}}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{ii}$$

where $F_0 = \frac{\phi_0}{g}$ is the force well under the pole.

Thus, giving different values to X, we obtain the corresponding values of x from i and of F from ii. In

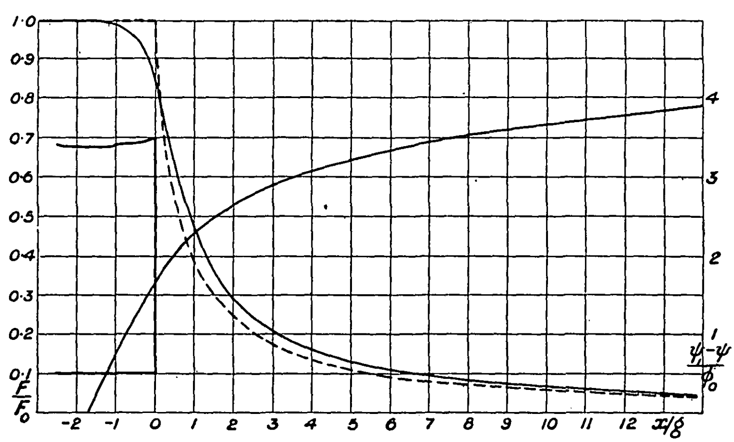

FIG. 3.

this way is obtained the Table I., and the full-line curve in Fig. 3. In the same figure is shown the curve,

TABLE I.

X	3	2½	1	0·75	0·5	0·25	0	−·25	−0·5	−·75	−1	−1·5	−2	−2·5	−3
x/ξ	−1·71	−1·08	−·42	−·235	−·064	0·131	0·338	0·583	0·861	1.202	1·61	2·78	4·65	7·72	12·7
F/F_o	0·999	0·991	0·945	0·904	0·855	0·791	0·705	0·617	0·518	0·429	0·335	0·218	0·136	0·082	0·05
$\frac{\psi_1-\psi_0}{\phi_0}$	0	0·638	1·28	1·44	1·59	1·95	1·91	2·07	2·23	2·39	2·55	2·87	3·19	3·51	3·82

(dotted), obtained in the usual manner, *i.e.*, by assuming that, when x is negative, the force $\frac{\phi_0}{g}$, and, when x is positive $\frac{\phi_0}{\frac{\pi}{2}x + g}$. A comparison of the curves shows that the latter assumption makes the force too great at the edge of the pole-piece, and too small at a little distance outside the edge. Messrs. Hawkins and Wightman's process of rounding the corner of the curve, besides being arbitrary, places the distance to which the rounding is to extend on a false basis, in expressing it as a fraction of the pole angle—of which it is quite independent. The distance in question should be expressed as a fraction of the length of the air-gap, and this fraction may be practically taken as unity, for Fig. 3 shows that at this distance under the pole the force has reached 99 per cent. of its greatest value.

Although, as above stated, the exact solution, in an actual case, is impossible, we may use the curve in Fig. 3 to correct the results of an approximate calculation, and so obtain results that are probably not greatly in error, in the following manner. Calculate the force in the usual way, *i.e.*, assume it constant under the pole, and inversely as $\xi x + g$ outside the pole, then multiply the results so obtained for any point by the ratio of the ordinate of the exact to that of the approximate curve in Fig. 3, at that point; or calling this ratio m, the force under the pole is proportional to $\frac{m}{g}$, and outside the pole to $\frac{m}{\xi x + g}$. A table of the values of m at different distances (x) from the pole tip is given below. (Table II.)

TABLE II.

x/g	−1	−0·75	−0·5	−0·25	0	0·25	0·5	0·75	1	1·5	2	3	4	6	8	10	12
m	0·99	0·973	0·952	0·91	0·84	1·04	1·14	1·20	1·22	1·21	1·20	1·18	1·16	1·13	1·09	1·05	1·01

In Fig. 3 is also shown the curve of total flux, to such a scale that the difference of ordinates at any two points gives the number of spaces equal to the air-gap that would be filled by the lines if they were concentrated to their maximum density. Thus the total flux from $x = -1\cdot71\,g$ to $x = 8g$ would fill $3\cdot54g$ with lines at maximum density, and if the lines had their maximum density throughout, the pole-piece would have to be widened by $(3\cdot54 - 1\cdot71) = 1\cdot83$ times the length of the air-gap. From this we get the permeance of the air-gap. This, however, neglects the effect of the adjacent pole, which causes the force to diminish to zero at the middle of the interpolar space. We might approxi-

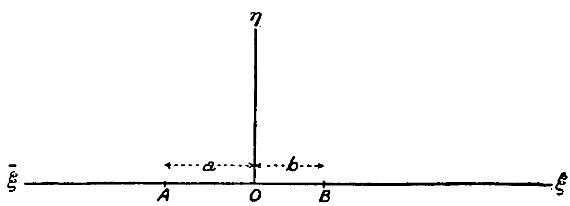

FIG. 4.—The Plane of $\xi\eta$.

mately take this into account by drawing the curves for the two poles, and adding their ordinates algebraically. This, however, is not a true solution of the problem presented by the two poles, and it would be well to examine how nearly it can be relied upon. As a matter of fact, it is possible to get an exact solution of the magnetic problem presented by two unlike rectangular poles and a straight armature, as follows:—

The transformation

$$z = \frac{g}{\pi} \log \left[\zeta + \frac{a-b}{2} + \sqrt{\overline{\zeta + a}\,\overline{\zeta - b}}\right] - \frac{c}{\pi} \sin^{-1}\left[\frac{2ab}{\overline{a+b}\,\zeta} - \frac{a-b}{a+b}\right]$$

(where c is a half of the distance between the poles $= g\sqrt{\frac{a}{b}}$ and the other letters are as before),

converts the half-plane, Fig. 4, into the region, Fig. 5, bounded by a pole-piece, armature, and the median line between the pole pieces, making the range of points ξ A O B ξ in Fig. 4 into those so marked in Fig. 5. If now we find a potential function which reduces to ϕ_0 on the pole-piece, and to zero on the armature and median line, we shall have solved the problem in hand. Such a function is given by

$$\zeta = e^{i\pi\frac{\phi + i\psi}{\phi_0}}$$

Substituting for ζ as before, and putting $\phi = 0,\ \pi\frac{\psi}{\phi_0} = \mathrm{X}$ we get

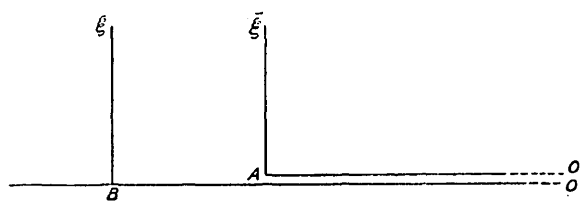

FIG. 5.—The Plane of $x\,y$.

$$\left\{\begin{aligned} x &= \frac{g}{\pi}\log\left[e^{-\mathrm{X}} + \frac{a-b}{2} + \sqrt{(e^{-\mathrm{X}}+a)(e^{-\mathrm{X}}-b)}\right] \\ &\quad - \frac{c}{\pi}\sin^{-1}\left(\frac{2ab}{a+b}e^{\mathrm{X}} - \frac{a-b}{a+b}\right) \qquad \ldots\ldots\ \text{iii} \\ \mathrm{F} &= \mathrm{F}_0\left(\frac{e^{-\mathrm{X}} - b}{e^{-\mathrm{X}} + a}\right)^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ \text{iv} \end{aligned}\right.$$

The origin in Fig. 5 is not yet determined. If we choose a and b so that

$$a + b = 2\,e^{-\frac{\pi}{2}\frac{c}{g}}$$

i.e. (since $\frac{a}{b} = \frac{c^2}{g^2}$), $a = \frac{2c^2}{c^2+g^2}\,e^{-\frac{\pi}{2}\frac{c}{g}}$, $b = \frac{2g^2}{c^2+g^2}\,e^{-\frac{\pi}{2}\frac{c}{g}}$,

we shall make $x = o$ at the edge of the pole-piece.

Fig. 6 shows the relation between F and x obtained from these formulæ (iii and iv), in three cases, viz., for $c = 2{\cdot}5g$, $c = 5g$, and $c = \infty$, the latter being the same curve as in Fig. 3. It will be seen that the second pole-piece has practically no effect at point under the first, thus showing that the process of superposing the effects of two such isolated poles as shown in Fig. 3 would lead to results far from the truth ; in fact, the closeness of the curves at their upper bend is their most noticeable feature. The curves for any other particular value of the ratio c/g can easily be traced from the above formulæ, though, if the curve $c = \infty$ be traced from Table I., the deviation therefrom might almost be drawn freehand with sufficient accuracy, after an

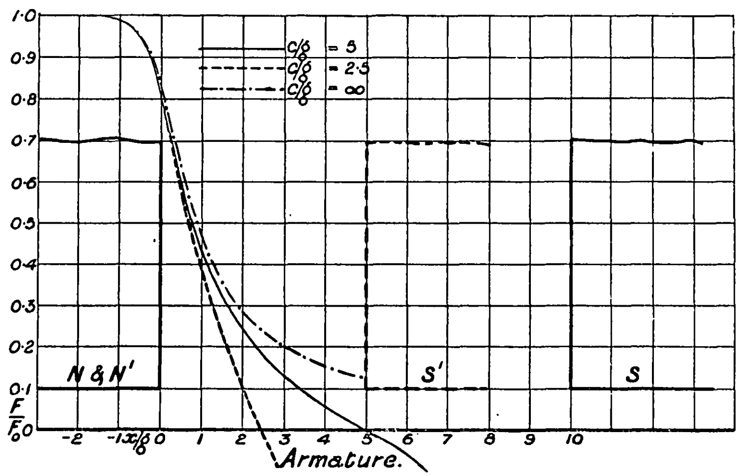

FIG. 6.

examination of Fig. 6. The slope of the curve where it cuts the armature is :

$$\frac{g}{F_0} \frac{dF}{dx} = \frac{\pi}{2} \cdot \frac{1}{\frac{c^2}{g^2} + 1}.$$

The permeance of the air-gap can now be obtained exactly. If we determine the total flux, up to $x = c$, *i.e.*, to half-way between the poles, and find what width we must add to the pole face to give the total flux this value, supposing the force to have its maximum value F_0, throughout the region underneath this increased pole, and to be zero elsewhere, then, expressing this increase in the usual manner, *i.e.*, as a fraction (λ) of the length of the air-gap, we shall find that λ is not a constant, as usually assumed, but is given by

$$\lambda = 0{\cdot}72 \log \left\{ \frac{1}{4}\left(1 + \frac{c^2}{g^2}\right) \right\} + \frac{1}{90}\left(\frac{c}{g} \tan^{-1} \frac{g}{c} \right),$$

where the logarithm is a common one, and the angle is expressed in degrees.

Table III. gives the value of λ for the different values of c/g within the possible limits in this subject.

TABLE III.

c/g	2	2·5	3	3 5	4	4·5	5	6	7	8	9	10	11	29
λ	0·68	0·79	0·90	0·99	1·07	1·15	1·22	1·33	1·42	1·51	1·58	1·65	1 7 0	..

The value of these calculations is well seen by comparing the values of λ in the above with those given by the usually assumed methods. Thus Messrs. Hawkins and Wightman obtain $\lambda = 1{\cdot}76$ for $c/g = 4$,[1] whilst the more nearly correct calculation gives the much smaller value $\lambda = 1{\cdot}07$; in fact, formulæ based on simple but wrong assumptions are dangerous things to use, and can rarely be relied on to give more than the order of the sought-for results.

I by no means recommend that one should go to the trouble of using these somewhat difficult formulæ in average practical cases, for they of course only apply strictly to the ideal cases considered, but submit that the reasons given at the beginning of this article are sufficient justification for placing on record the solution of a problem which more nearly resembles the actual problem of the fringe of lines in the interpolar space than any which has, as far as I am aware, been exactly worked out.

THE RELUCTANCE OF THE TEETH IN A SLOTTED ARMATURE.

By W. B. HIRD, B.A., Member.

In predetermining the characteristic of a dynamo with slotted armature the calculation of the ampere-turns required

[1] Page 449.