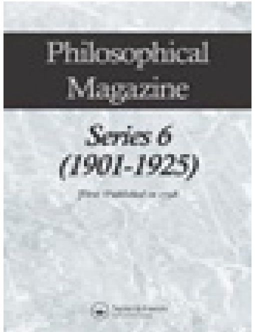

Taylor & Francis
Taylor & Francis Group

Philosophical Magazine Series 6

ISSN: 1941-5982 (Print) 1941-5990 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm17

XX. Note on the potential of a symmetrical system

T.J. I'A. Bromwich

In taking my experiments one by one, it is possible to point out in each some defect which might account for the negative result of that one experiment. But the general conclusions which I have stated * are based on the *ensemble* of four widely different experiments, and are confirmed by a fifth, a very conclusive one, regarding the existence of open currents.

Each of these experiments has, besides, been subjected to numerous criticisms. Up to the present I have always met a criticism by an experiment, and this latter has always confirmed my conclusions.

While thanking Mr. Wilson for drawing my attention to a point in connexion with my first experiment which might have been unnoticed by me, I think myself justified in replying to him that :—

(1) Electric convection produces no magnetic effect.

(2) There is no electrostatic effect on a charged conductor due to a variable magnetic field.

I cannot conclude without tendering to Professor H. Poincaré my best thanks for the suggestions received from him in the preparation of this brief note.

XX. *Note on the Potential of a Symmetrical System.* *By* T. J. I'A. BROMWICH †.

IT was proved by Legendre that if the potential of a system (symmetrical about Oz) is known at all points of the axis of z, then the value of the potential can be expressed at any point of space in terms of zonal harmonics. But it does not seem to have been remarked that this method may lead in some cases to an apparent discontinuity in the potential functions when so expressed.

To illustrate the point, let us examine the potential of a circular disk for all points of space; this is, of course, a stock example of Legendre's method, given in all the ordinary text-books. From Thomson and Tait's 'Natural Philosophy' (1890 edition), Art. 546, we find

$$V_0 = 2\pi\rho\left(c_1\frac{a^2}{r} + c_2\frac{a^4}{r^3}P_2 + c_3\frac{a^6}{r^5}P_4 + \dots\right) \qquad (r>a)$$

$$V_1 = 2\pi\rho\left(a - rP_1 + c_1\frac{r^2}{a}P_2 + c_3\frac{r^4}{a^3}P_4 + \dots\right) \quad \left(\begin{matrix} r<a \\ 0<\theta<\frac{1}{2}\pi\end{matrix}\right)$$

or $$2\pi\rho\left(a + rP_1 + c_1\frac{r^2}{a}P_2 + c_3\frac{r^4}{a^3}P_4 + \dots\right), \quad \left(\begin{matrix} r<a \\ \frac{1}{2}\pi<\theta<\pi\end{matrix}\right)$$

* V. Crémieu, *Thèse de Paris,* Gauthier-Villars, 1901.

† Communicated by the Author.

where ρ is the surface-density, a is the radius of the disk, the origin is the centre of the disk and its plane is the plane of xy. Further,

$$r^2 = x^2 + y^2 + z^2, \quad z = r\cos\theta,$$

and P_n stands for $P_n(\cos\theta)$, Legendre's coefficient of order n; also c_n is the coefficient of x^n in the expansion of $(1+x)^{\frac{1}{2}}$ in powers of x, so that

$$(1+x)^{\frac{1}{2}} = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots.$$

For the future we shall write $\cos\theta = \mu$, for brevity.

At the plane of the disk ($z=0$) it is easy to see that the two values of V_1 are continuous, but that $\frac{\partial V_1}{\partial z}$ is discontinuous; a fact which agrees with what we know from the general properties of the potential. But, apparently, at $r=a$ V_0 is not equal to V_1; and this is the point which I wish to clear up, for, of course, there can be no discontinuity in the potential and its differential coefficients at any point in free space. From the previous results we have at $r=a$,

$$V_0 - V_1 = 2\pi\rho a[(c_1-c_0) + P_1 + (c_2-c_1)P_2 + \dots + (c^{n+1}-c_n)P_{2n} + \dots], \quad (\mu>0)$$

or

$$2\pi\rho a[(c_1-c_0) - P_1 + (c_2-c_1)P_2 + \dots + (c_{n+1}-c_n)P_{2n} + \dots]. \quad (\mu<0)$$

Hence, if there is to be no discontinuity, remembering that $P_1 = \mu$, we must have

$$+\mu = (c_0-c_1) + (c_1-c_2)P_2 + \dots + (c_n-c_{n+1})P_{2n} + \dots, \quad (\mu>0)$$

$$-\mu = (c_0-c_1) + (c_1-c_2)P_2 + \dots + (c_n-c_{n+1})P_{2n} + \dots \quad (\mu<0)$$

In order to test this, let us expand $f(\mu)$ in terms of Legendre's coefficients, where

$$f(\mu) = +\mu \quad (\mu>0)$$

and

$$= -\mu \quad (\mu<0).$$

We know that, with certain restrictions on the nature of $f(\mu)$, of the same type as Dirichlet's conditions for Fourier's series, we can write

$$f(\mu) = \sum_{n=0}^{\infty} A_n P_n(\mu),$$

where

$$A_n = \tfrac{1}{2}(2n+1)\int_{-1}^{+1} f(\mu) P_n(\mu) d\mu.$$

Here the necessary conditions for the expansion are satisfied, and we find

$$A = \tfrac{1}{2}(2r+1)\left[\int_{-1}^{0}(-\mu)P_r(\mu)d\mu + \int_{0}^{1}(+\mu)P_r(\mu)d\mu\right],$$

i. e.

$$A_r = (2r+1)\int_0^1 \mu P_r(\mu)d\mu, \qquad (r \text{ even})$$

$$= 0. \qquad (r \text{ odd})$$

Now

$$(2r+1)\mu P_r(\mu) = (r+1)P_{r+1}(\mu) + rP_{r-1}(\mu),$$

and so, if $r = 2n$,

$$A_{2n} = \int_0^1 [(2n+1)P_{2n+1}(\mu) + 2nP_{2n-1}(\mu)]d\mu.$$

It is easy to prove that

$$\int_0^1 d\mu\left[1 + \sum_{r=1}^{\infty} h^r P_r(\mu)\right] = \int_0^1 \frac{d\mu}{(1-2\mu h + h^2)^{\frac{1}{2}}}$$

$$= \frac{1}{h}[(1+h^2)^{\frac{1}{2}} - (1-h)] = 1 + \sum_{n=1}^{\infty} c_n h^{2n-1},$$

and so

$$\int_0^1 P_{2n-1}(\mu)d\mu = c_n.$$

This gives

$$A_{2n} = (2n+1)c_{n+1} + 2nc_n.$$

But from the definition of the c's it follows that

$$\frac{c_{n+1}}{c_n} = \frac{\frac{1}{2} - n}{n+1} = -\frac{2n-1}{2(n+1)},$$

and thus

$$A_{2n} = (c_n - c_{n+1}) + [(2n-1)c_n + 2(n+1)c_{n+1}],$$

$$= c_n - c_{n+1}.$$

Also

$$A_0 = \tfrac{1}{2}\int_{-1}^{+1} f(\mu)d\mu = \int_0^1 \mu d\mu = \tfrac{1}{2} = c_0 - c_1.$$

It follows that the expansion

$$(c_0-c_1)+(c_1-c_2)P_2+\ldots+(c_n-c_{n+1})P_{2n}+\ldots$$

has the value $+\mu$ if $\mu>0$, and the value $-\mu$ if $\mu<0$. Consequently $V_0=V_1$ at $r=a$, by what has been explained before.

Consider next the value of $\left(\frac{\partial V_1}{\partial r}-\frac{\partial V_0}{\partial r}\right)$ at $r=a$; this should also vanish, since there is no surface-density on the sphere. We find that its value is

$$2\pi\rho[-\mu+c_1+(2c_1+3c_2)P_2+\ldots$$
$$+\{2nc_n+(2n+1)c_{n+1}\}P_{2n}+\ldots] \qquad (\mu>0),$$

or

$$2\pi\rho[+\mu+c_1+(2c_1+3c_2)P_2+\ldots$$
$$+\{2nc_n+(2n+1)c_{n+1}\}P_{2n}+\ldots]. \qquad (\mu<0)$$

Each of these expressions vanishes, according to the value found above for A_{2n}.

Hence

$$\frac{\partial V_1}{\partial r}=\frac{\partial V_0}{\partial r} \quad \text{at } r=a.$$

Now V_1, V_0 satisfy the same differential equation of the second order (Laplace's),

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0,$$

and, at $r=a$, $V_0=V_1$ and $\frac{\partial V_0}{\partial r}=\frac{\partial V_1}{\partial r}$ for all values of θ between 0 and π. It follows that V_0 must be the analytical continuation of V_1 beyond the sphere $r=a$; and the discontinuity at $r=a$ is only apparent, not real. A similar point occurs in connexion with the magnetic potential of a circular coil, carrying an electric current; the expressions for this are given in the same article of Thomson and Tait. The numerical details are slightly different, but the principle involved is exactly the same as in the above work.

S. John's College, Cambridge.
26th June, 1901.