Final results.—The departure of the curves from a straight horizontal line shows the presence of overtones as well as the prime tone; while the fact that the curves appear to asymptote to this line shows that the overtones die away quicker than the fundamental.

By both methods of experiment we see that the two curves agree best when $p=\cdot 8$; then substitute this value in (9) and we get a value for the frequency of vibration such that

$$\frac{N_1}{N}=\cdot 986.$$

I hope to make further experiments on tuning-forks or bars of acoustic frequency. I desire to render my best thanks to one of my colleagues, Dr. E. H. Barton, for his kind help in the work.

University College, Nottingham,
July 14, 1904.

LVIII. *The Limits of Economy of Material in Frame-structures.* *By* A. G. M. Michell, *M.C.E., Melbourne**.

MAXWELL has shown† that for all frames under a given system of applied forces

$$\Sigma\,.\,l_p f_p-\Sigma\,.\,l_q f_q=C, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where f_p is the tension in any tie-bar of length l_p, f_q the thrust in any strut of length l_q, and the first sum is taken for all the ties, the second for all the struts. C is shown to be a function of the applied forces and the coordinates of their points of application, and independent of the form of the frame.

Starting from this result, we can find in certain cases lower limits to the quantity of material necessary to sustain given forces, and also assign the forms of frames which attain the limit of economy.

If the greatest tensile stress allowable in the material which is to be employed is P, and the greatest compressive stress Q, the least volume of material in a given frame, consistent with security, is

$$\Sigma\,.\,l_p\frac{f_p}{P}+\Sigma\,.\,l_q\,.\,\frac{f_q}{Q}=V. \quad . \quad . \quad . \quad . \quad (2)$$

* Communicated by the Author.

† Scientific Papers, ii. pp. 175–177.

Now, considering a number of different frames in equilibrium under the same set of external forces, in that one of the frames in which V is least

$$2PQ\,.\,V+(P-Q)C \quad \text{is also least;}$$

$$\textit{i. e.,}\ 2PQ\left\{\Sigma\,.\,l_p\frac{f_p}{P}+\Sigma\,.\,l_q\frac{f_q}{Q}\right\}+(P-Q)\{\Sigma\,.\,l_pf_p-\Sigma\,.\,l_qf_q\}$$

$$=(P+Q)\{\Sigma\,.\,l_pf_p+\Sigma\,.\,l_qf_q\} \quad \text{is least,}$$

$$\text{or} \qquad \Sigma\,.\,l\,[f] \text{ is least,} \qquad \qquad (3)$$

where $[f]$ denotes the numerical value of the force in any bar, and the sum is taken for both struts and ties.

One proof given by Maxwell of the equation (1) above depends on consideration of the virtual work of the applied forces and internal stresses during an imposed uniform dilatation or contraction of the frame. Consideration of a more general type of imposed deformation will furnish information as to the quantity $\Sigma\,.\,l\,[f]$.

Let the space within a given boundary, which encloses a number of different frames subjected in turn to the same set of applied forces, undergo an arbitrary deformation of which the frames partake and such that no linear element in the space suffers an extension or contraction numerically greater than $\epsilon\,.\,\delta l$, where δl is the length of the element and ϵ a given small fraction.

In this deformation, let any bar of length l of one of the frames, A, undergo the small change of length el, which is to be taken as positive when it increases the existing strain in the bar due to the applied forces, negative in the contrary case. The increase in the elastic energy of the bar is elf, and for the whole frame

$$\Sigma\,.\,elf=\delta W,$$

by the principle of Virtual Work, where δW is the virtual work of the applied forces, and is independent of the form of the frame A.

$$\text{Thus} \qquad \delta W=\Sigma\,.\,elf \ngtr \Sigma\,.\,[e]l[f]$$

$$\ngtr \Sigma\,.\,\epsilon l\,[f] \ngtr \epsilon\Sigma\,.\,l\,[f]$$

$$\text{or} \qquad \Sigma\,.\,l_A\,[f]_A \nless \frac{\delta W}{\epsilon},$$

indicating by the suffixes that the inequality applies to the frame A.

If, however, a frame, M, can be found, such that all its parts have their strains increased equally and by as much as any elementary line in the deformed space, *i. e.* if $e=\epsilon$ in all parts, the signs of inequality may be replaced by that of equality, and

$$\Sigma . l_M [f]_M = \frac{\delta W}{\epsilon} \ngtr \Sigma . l_A f_A,$$

so that $\Sigma . l_M [f]_M$ is a minimum, and consequently from (3), V_m, the volume of material in the frame M, is also a minimum.

A frame therefore attains the limit of economy of material possible in any frame-structure under the same applied forces, if the space occupied by it can be subjected to an appropriate small deformation, such that the strains in all the bars of the frame are increased by equal fractions of their lengths, not less than the fractional change of length of any element of the space.

If the space subjected to the deformation extends to infinity in all directions, the volume of the frame is a minimum relatively to all others, otherwise it will have been shown to be a minimum only relatively to those within the same assigned finite boundary.

The condition $e=\epsilon$ can evidently be satisfied when all the bars of a frame have stresses of the same sign. The test deformation to be applied is then a uniform dilatation or contraction (according as the frame is in tension or compression) of a space enclosing the frame and extending to any finite boundary or to infinity at pleasure.

The simplest minimum frames of this special class are:—

I. Ties and struts subjected to a single pair of equal and opposed forces.

II. Triangular and tetrahedral frames under forces applied at the angles of the figure, and acting along lines which intersect within the figure.

III. Catenaries in general, the points of application of the applied forces, as well as the forces themselves, being given.

In all these cases the minimum volume of the frame is

$$V_m = \frac{\Sigma . l[f]}{P} = \frac{\delta W}{\epsilon P} = \frac{\Sigma . \epsilon F r \cos\theta}{\epsilon P}$$

$$= \frac{\Sigma . F r \cos\theta}{P},$$

where F is one of the applied forces, r the distance of its point of application, R, from an arbitrary fixed point O,

θ the angle between RO and the direction of F, and P is the allowable stress in the material.

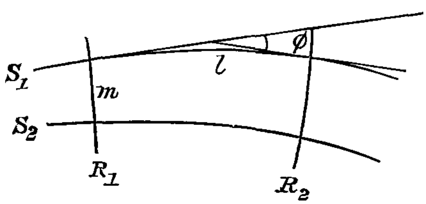
A more general class of frames satisfying the condition $e=\epsilon$, consists of those whose bars, both before and after the appropriate deformation, form curves of orthogonal systems.

In such a system the test strains may be equal and of the same sign, or equal and opposite, in directions at right angles, without the strains in the bars being exceeded by those of any other lines in the field. In the first case the strain is evidently the same in all directions; in the second, if λ, $-\lambda$ are the strains in the principal directions, the strain in any direction at an angle θ to one of them is $\pm\lambda \cos 2\theta=\lambda'$,

$$\therefore [\lambda'] \ngtr [\lambda].$$

The condition that a two-dimensional orthogonal system shall remain orthogonal after equal and opposite extensions of its two series of curves, is that the inclination between any two adjacent curves of the same series is constant throughout their length. This is easily seen as follows:—

Let l be the elementary length, ϕ the change of direction of the curve S_1, between its points of intersection with two



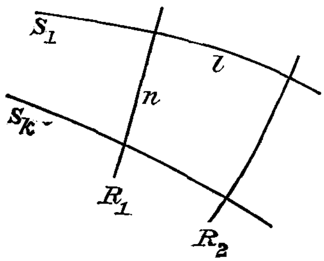
adjacent curves R_1R_2 of the other series, and let m be the elementary length of R_1 between the two curves S_1, S_2 of the S-series, m being taken small in comparison with l.

Let the curves of the S-series be extended by the small fraction λ of their length.

If the element remains rectangular the change in the angle ϕ must be equal to the change in the inclination of the elements R_1 and R_2 to each other,

$$\therefore \delta\phi=\frac{\lambda l-\lambda(l+\frac{dl}{dm}\cdot m)}{m}=-\lambda\frac{dl}{dm}=\lambda\phi.$$

Take now a new element, with sides of comparable length, l, n, formed by the pairs of curves R_1R_2, S_1S_k. Let ψ be the change of direction of R_1 between S_1 and S_k, and let λ' be the extension of the R curves.



Then, as before, $\delta\psi = \lambda'\psi$; and making a circuit around the element, since the angles remain right angles,

$$\lambda \, . \frac{d\phi}{dn} n - \lambda' \, . \frac{d\psi}{dl} \, . \, l = 0,$$

but $\frac{d\phi}{dn} \, . \, n - \frac{d\psi}{dl} \, . \, l = 0$, because the angles were right angles before the deformation. Thus since $\lambda' \neq \lambda$, by assumption

$$\frac{d\phi}{dn} = 0, \quad \text{and} \quad \frac{d\psi}{dl} = 0. \qquad \text{Q.E.D.}$$

There are two general classes of orthogonal curves satisfying the required conditions, viz. :—

I. Systems of tangents and involutes derived from any evolute curves.

(Since such systems are bounded by the evolute curves, the corresponding frames are in general of minimum volume only relatively to others with the same finite boundaries.)

II. Orthogonal systems of equiangular spirals, with systems of concentric circles and their radii, and rectangular networks of straight lines, as special cases.

Frames whose bars coincide with the curves of any of these systems are therefore frames of minimum quantity of material, for any system of forces consistent with their equilibrium and continuity of displacement.

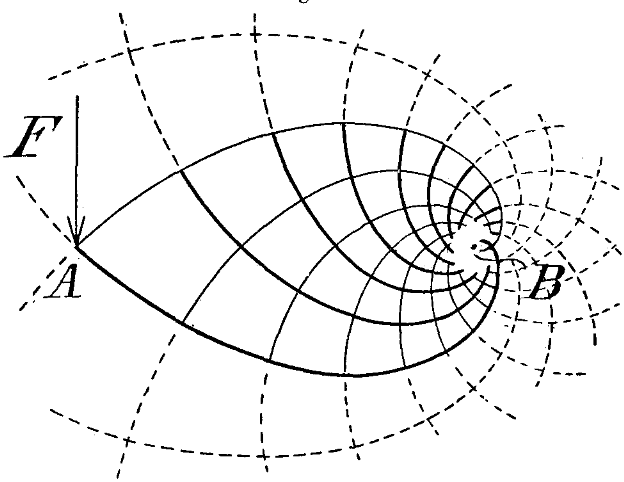
Frames may also be built up of parts of different systems of these classes provided that continuity of displacement is secured along the lines of junction.

Examples of such frames of minimum quantity of material, for some elementary systems of forces, are given in the accompanying figures.

Compression bars in each case are indicated by thick, and tension bars by fine, continuous lines. Those portions of the lines of principal strain on which material bars are not required are shown by dotted lines.

1. (Fig. 1.) A single force F applied at A, and acting at right angles to the line AB, is balanced by an equal and opposite force and a couple, of moment $F \times AB$, applied at B.

Fig. 1.



The minimum frame is formed of two similar equiangular spirals having their origin at B and intersecting orthogonally at A, together with all the other spirals orthogonal to these and enclosed between them.

All the spirals of one series are equally extended, those of the other system equally compressed to the same degree. The spirals may be extended to infinity, and the infinite field will remain continuous with equal and opposite strains in the principal directions at all points. The necessary volume of material is infinite if the frame is continued to a mere point at B; but if forces equivalent to the force F, and the couple $F \times AB$ at B, are distributed over a small circle with its centre at B, and of radius r_0, the volume necessary is

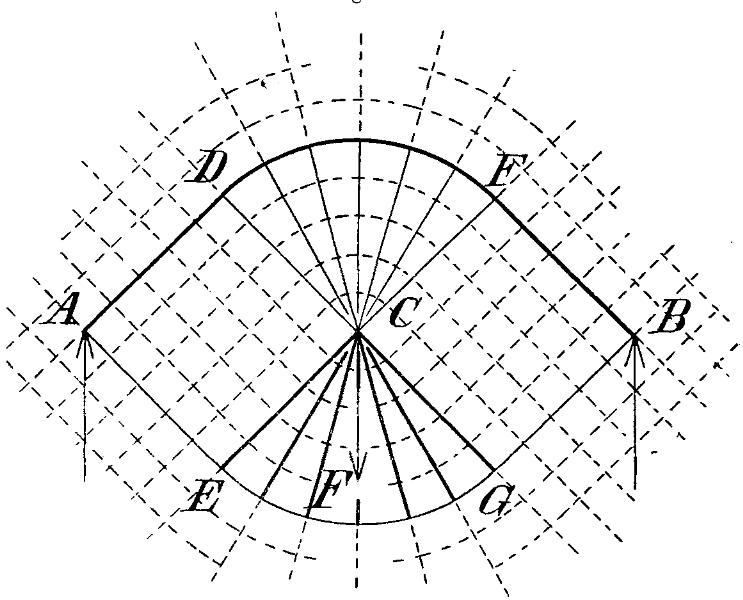
$$Fa \, . \log \frac{a}{r_0} \cdot \left(\frac{1}{P} + \frac{1}{Q}\right),$$

where $a = AB$, and P, Q are the stresses allowed for tension and compression.

2. *Centrally-loaded Beam.*—(Fig. 2.) A single force, F, is applied at C, the middle point of the line AB, and is balanced by equal parallel forces at A and B.

The minimum frame is composed of the two quadrantal bars DF and EG, having their common centre at C, of all

Fig. 2.



the radii of these quadrants, and their four tangents AD, AE, BF, and BG. The orthogonal system is completed and extended to infinity as indicated by the dotted lines.

The quadrants above C and all straight lines extending obliquely downwards and outwards from the line of action of F are compressed, all the other lines extended.

The necessary volume of material is

$$Fa\left(\frac{1}{2}+\frac{\pi}{4}\right)\left(\frac{1}{P}+\frac{1}{Q}\right),$$

where a=AC=CB.

2*a*. (Fig. 3.) With the same applied forces the condition may be imposed that the frame shall lie wholly on one side of the line AB.

The form of the minimum frame is then as in fig. 3, and consists of the semicircle on AB and its radii. The figure to be subjected to the test deformation consists of the semi-infinite plane below AB, all the radii from C being extended and the semicircles about C compressed.

The minimum volume of the frame is

$$\mathrm{F}a\,.\,\frac{\pi}{2}\left(\frac{1}{\mathrm{P}}+\frac{1}{\mathrm{Q}}\right),$$

and therefore exceeds the frame of fig. 2, in which no limitation of extent was imposed, in the ratio $2\pi : (2+\pi)$, or 1·22 : 1.

Fig. 3.

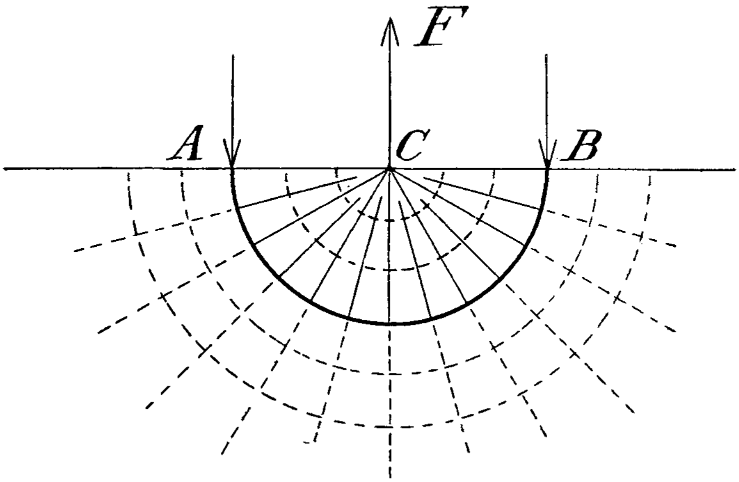
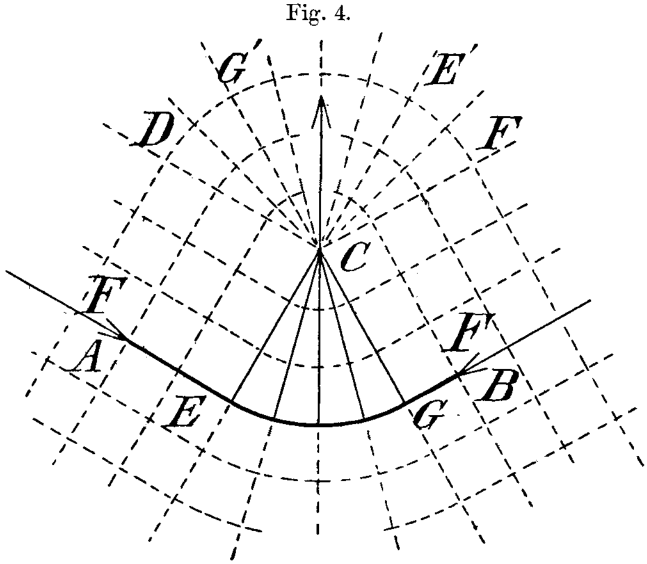


Fig. 4.



3. (Fig. 4.) Constructions similar to that of figs. 2 & 3 give the minimum frames for three forces, two of which are

equal and directed to a point on the line of action of the third outside the triangle formed by the points of application of the forces. Fig. 4 is an example.

The sectors DCG′, FCE′ undergo uniform bulk compression, the remainder of the field pure shear as in fig. 2.

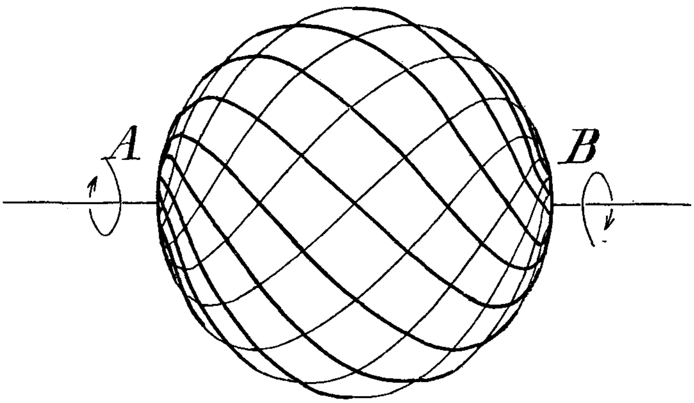
The volume of the frame is

$$F \times \left\{ CE \times \alpha \times \left(\frac{1}{P} + \frac{1}{Q}\right) + (AE + BG) \times \frac{1}{Q} \right\},$$

where α is the angle ECG.

4. (Fig. 5.) Equal and opposite couples applied at points A, B on the straight line AB.

Fig. 5.



The minimum frame consists of the series of rhumb-lines inclined at 45° to the meridians of the sphere having its poles at A and B.

All the rhumb-lines of one series and all similar lines on spheres concentric with the spherical frame are uniformly and equally extended, all the lines orthogonal to these on the spheres equally compressed.

The third system of orthogonals, viz., radii from the centre of the sphere, are unchanged in length and simply rotated about the axis AB.

The minimum volume of the frame is

$$\frac{2L}{a} \log \tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right) \times \left(\frac{1}{P} + \frac{1}{Q}\right),$$

where L is the moment of the transmitted couple, $2a = AB$, and λ is the latitude of small circles about each pole corresponding to the circle of radius r_0 in Case 1.

Melbourne, April 19, 1904.