And ΩT, $\Omega' T'$ intersect on the join of O (the incentre) and the Kiepertian point.

The equation to TT' is

$$a\alpha(a^2-bc)+\ldots+\ldots=0;$$

to the join of O and mid-point of TT' is

$$a\alpha(b-c)+\ldots+\ldots=0,$$

which passes through G; hence G is centroid of $\Delta OTT'$.

The Isostereans are easily obtained thus: let l line from A, meet $\beta_2 T\gamma_1$ in h; then we have

$$\gamma_1 h = c^2/2s = B\gamma_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

and therefore Bh is bisector of $\angle B$.

[*Note.*—I began the study of these lines with considering the general case of the transversals obliquely drawn, but I have not obtained many simple results. I hope, however, to return to this view of the question as it was suggested in Mr. Kempe's remarks when my communication was made to the Society.]

Symmetric Functions and the Theory of Distributions.

By Captain P. A. MacMahon, R.A.

[*Read March 8th,* 1888.]

The theory of distributions is discussed in an elementary manner in Whitworth's *Choice and Chance*, Third Edition, Ch. III. The subject is studied in France under the title *L'Analyse Combinatoire.* There have been very few researches during recent years, and none, so far as my knowledge extends, which proceed by the method employed in this paper. This method is essentially constructive in its nature.

The investigation has for its object the bringing forward of the theory as an analytical weapon of considerable power in algebraical research. The notation employed is new, and possesses the advantage of being the simplest that it is possible to use.

Among results of minor importance and interest, four important and very general purely algebraical theorems are established. These are—

(1) A comprehensive law of algebraic reciprocity.

(2) A cardinal theorem of symmetric function expressibility.

(3) A generalisation of Vandermonde's (or Waring's) formula in symmetric functions.

(4) The formation of symmetrical symmetric-function tables corresponding to every partition of every number.

The research is continued, from the point of view of the Algebra of Symmetric Functions, in a paper by the author ("Memoir on a New Theory of Symmetric Functions"), which will shortly appear in No. 4, Vol. x. of the *American Journal of Mathematics.*

The notation employed throughout is that of partitions.

The Theory of Partitions, from the point of view of the Theory of Numbers, has been studied chiefly by Cayley, Sylvester, Glaisher, Franklin, and Hammond. These researches have appeared principally in the *Philosophical Transactions of the Royal Society*, the *American Journal of Mathematics*, the *Quarterly Journal of Mathematics*, and the *Messenger of Mathematics.*

An important reference is "A constructive Theory of Partitions," by J. J. Sylvester, *American Journal of Mathematics*, Vol. v., p. 251.

The first mathematician who employed the notation of a partition in ordinary algebra was Meyer Hirsch in his Algebra published in 1812; since then the idea has been further developed by Cayley, Hammond, the author of this paper, and probably a few others. The following memoirs may be consulted:—

Cayley: "A Memoir on the Symmetric Functions," *Phil. Trans. R. S.*, 1857.

The Author: "Seminvariants and Symmetric Functions," *Amer. Jour. of Math.*, Vol. vi.; the Author: "On Perpetuants," *Amer. Jour. of Math.*, Vol. vii.; the Author: "Memoir on Seminvariants," *Amer. Jour. of Math.*, Vol. viii.; Hammond: "On Perpetuants," *Amer. Jour. of Math.*, Vol. viii.; the Author: "The Expression of Syzygies," &c., *Amer. Jour. of Math.*, Vol. x.

Preliminary.

As defined by Whitworth (*loc. cit.*), "Distribution" is the separation of a series of elements into a series of classes; in the general problem, the things to be distributed may be of any species, viz., there may be n things, of which p are of one kind, q of a second kind, r of a third, &c. ..., where $p+q+r+ \ldots = n$; it is then convenient to speak of things or objects $(pqr \ldots)$ where, in this particular connection, the partition $(pqr \ldots)$ is to be regarded as defining the objects in regard to species; again, the classes into which the objects are to be distributed may be of any species, and this leads us to speak of classes $(p_1 q_1 r_1 \ldots)$, where $p_1+q_1+r_1+\ldots = n_1 =$ the number of classes; the partition $(p_1q_1r_1 \dots)$ here defines the classes in regard to species, indicating p_1 classes of one description, q_1 of a second, r_1 of a third, and so forth.

It should be observed that, in the use of partitions, repetitions of the same part are indicated by an index; for instance

$$(pppqqr \dots) \text{ is written } (p^3q^2r \dots).$$

If no attention is paid to the order of the objects (whatever be their species) in a class, the distribution may be described as one into "parcels"; each parcel is a class of unarranged objects.

If, however, permutations are permissible amongst objects in the same class, the distribution is said to be one into "groups"; each group is a class of arranged objects.

Two chief problems may be enunciated as follows:—

"To determine the number of distributions of objects $(pqr \dots)$ into parcels $(p_1q_1r_1 \dots)$."

"To determine the number of distributions of objects $(pqr \dots)$ into groups $(p_1q_1r_1 \dots)$."

Further, we may discuss each of these problems when the distributions are subject to certain restrictions; it is from the consideration of restricted distributions that most of the analytical results of this paper are evolved.

SECTION 1.

The Distribution Function.

Let $\alpha, \beta, \gamma, \dots$ be the roots of the equation

$$x^n - a_1x^{n-1} + a_2x^{n-2} - \dots = 0.$$

The symmetric function $\Sigma\alpha^p\beta^q\gamma^r \dots$, where $p+q+r+\dots = n$, is, in the partition notation, written

$$(pqr \dots).$$

Let
$$A_{(pqr\dots),(p_1q_1r_1\dots)}$$

denote the number of ways of distributing objects, defined by the partition $(pqr \dots)$, into parcels, defined by the partition $(p_1q_1r_1 \dots)$. I suppose there to be m parcels, so that $p_1+q_1+r_1+\dots = m$.

It will be convenient henceforward to speak simply of the distribution of objects $(pqr \dots)$ into parcels $(p_1q_1r_1 \dots)$.

I attach the number

$$A_{(pqr\dots),(p_1q_1r_1\dots)}$$

to the symmetric function

$$(pqr \dots),$$

and construct the expression

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

by taking the summation over every partition $(pqr\ldots)$ of the number n.

Definition. The Distribution Function of n objects into parcels $(p_1q_1r_1\ldots)$ is the expression

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

where $$p+q+r+\ldots = n.$$

I write also

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots) = Dp(n),\ (p_1q_1r_1\ldots).$$

Let, also,

$$B_{(pqr\ldots),(p_1q_1r_1\ldots)}$$

denote the number of ways of distributing objects $(pqr\ldots)$ into groups $(p_1q_1r_1\ldots)$.

Definition. The Distribution Function of n objects into groups $(p_1q_1r_1\ldots)$ is the expression

$$\Sigma B_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

where $$p+q+r+\ldots = n.$$

In this case I write

$$\Sigma B_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots) = Dg(n),\ (p_1q_1r_1\ldots).$$

My present purpose is the study of these two Distribution Functions.

Section 2.

Parcels, m in number (i.e., $m = n$).

Let h_s be the homogeneous product-sum, of degree s, of the n quantities $\alpha, \beta, \gamma, \ldots$; so that

$$h_1 = \Sigma\alpha = (1),$$
$$h_2 = \Sigma\alpha^2+\Sigma\alpha\beta = (2)+(1^2),$$
$$h_3 = \Sigma\alpha^3+\Sigma\alpha^2\beta+\Sigma\alpha\beta\gamma = (3)+(21)+(1^3),$$
&c. = &c.

Consider the product $h_{p_1}h_{q_1}h_{r_1}\ldots$.

The symmetric function $(pqr\,...)$ will, on performing the multiplication, be produced a certain number of times. In the factor h_{p_1} every term is of degree p_1 in the quantities. Taking any particular term, write down the p_1 quantities occurring therein in any order with a dot between each pair of consecutive quantities. We may consider these p_1 quantities as distributed into p_1 similar parcels, one quantity into each parcel. In the same way, any q_1 quantities which occur in any term of h_{q_1} may be considered to be q_1 quantities distributed into q_1 parcels, similar to one another, but different from the former. Hence it is clear that the number of times that the symmetric function $(pqr\,...)$ occurs in the development of the product $h_{p_1}h_{q_1}h_{r_1}\,...$ is precisely the number of ways that it is possible to distribute objects $(pqr\,...)$ into parcels $(p_1q_1r_1\,...)$, one object in each parcel. Hence, when $m = n$, and no parcel is empty,

$$Df(n),\ (p_1q_1r_1\,...) = \Sigma A_{(pqr...),\,(p_1q_1r_1...)}\,(pqr...) = h_{p_1}h_{q_1}h_{r_1}\,...\,.$$

Consider, for a moment, the distribution of objects (43) into parcels (52), and represent objects and parcels by small and capital letters respectively. One distribution is represented by the scheme

$$\begin{matrix} A & A & A & A & A & B & B \\ a & a & a & a & b & b & b \end{matrix}$$

wherein an object denoted by a small letter is placed in a parcel denoted by the capital letter directly above it. Corresponding to this distribution of objects (43) into parcels (52), we have a distribution of objects (52) into parcels (43), given by the scheme

$$\begin{matrix} A & A & A & A & B & B & B \\ a & a & a & a & a & b & b \end{matrix}$$

derived from the former by interchanging rows as well as small and capital letters. The process is clearly general and exhibits a one-to-one correspondence between the distributions of objects $(pqr\,...)$ into parcels $(p_1q_1r_1\,...)$, and the distributions of objects $(p_1q_1r_1\,...)$ into parcels $(pqr\,...)$. It is, in fact, an intuitive observation, that we may either consider an object placed in or attached to a parcel, *or* a parcel placed in or attached to an object.

Hence the very important theorem

$$A_{(pqr..),\,(p_1q_1r_1...)} = A_{(p_1q_1r_1...),\,(pqr..)}.$$

Analytically this result leads to a law of algebraic symmetry which I now enunciate.

Theorem.—" The coefficient of symmetric function $(pqr\,...)$ in the development of the product $h_{p_1}h_{q_1}h_{r_1}\ldots$ is equal to the coefficient of symmetric function $(p_1q_1r_1\ldots)$ in the development of the product $h_ph_qh_r\ldots$."

This law of symmetry I established in the *Quarterly Journal of Mathematics.*

The problem of the distribution of n objects into n parcels, one object into each parcel, is thus completely solved by means of a table of symmetric functions which expresses the h-products as linear functions of the single partition forms. (*Vide* the Tables at the end of the paper.)

SECTION 3.

Parcels of species (1^m), *where* $m<n$.

I now discuss the distributions of n objects into m parcels, no two of which are similar. Whitworth would describe the problem as a distribution into m DIFFERENT parcels.

Let $\qquad (p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots), \qquad [\Sigma\pi = m,\ \Sigma\pi p = n],$

be any partition of n into m parts.

Of the whole number of distributions, there will be a certain number such that π_s parcels each contain p_s objects,

$$(s = 1, 2, 3, \ldots).$$

The distribution function of this particular case of the distribution is

$$\frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\ldots}\,h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}h_{p_3}^{\pi_3}\ldots.$$

To see how this is, observe that the product $h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}h_{p_3}^{\pi_3}\ldots$ is susceptible of $\frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\ldots}$ permutations. The parcels are all different, and hence there are distributions corresponding to each of these permutations. By the last section, for each of these permutations there will be a distribution function

$$h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}h_{p_3}^{\pi_3}\ldots,$$

and for the aggregate of permutations a distribution function

$$\frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\ldots}\cdot h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}h_{p_3}^{\pi_3}\ldots.$$

Hence the distribution function of n objects into parcels (1^m) is

$$\Sigma \frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3}\ldots, \qquad [\Sigma\pi = m,\ \Sigma\pi p = n],$$

that is, it is the coefficient of x^n in the expansion of

$$(h_1x+h_2x^2+h_3x^3+\ldots)^m.$$

We may write this result

$$Dp\,(n),\,(1^m) = \Sigma A_{(pqr\ldots),\,(1^m)}\,(pqr\ldots) = \Sigma \frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3}\ldots,$$

where $\Sigma\pi = m,\quad \Sigma p\pi = n.$

Section 4.

General value of $A_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),\,(1^m)}$.

We require the coefficient of x^n in the expansion of

$$(h_1x+h_2x^2+h_3x^3+\ldots)^m = u^m, \text{ suppose.}$$

Put $\qquad f(x) = 1+h_1x+h_2x^2+h_3x^3+\ldots = 1+u,$

then $\qquad (1+u)^m = (1-\alpha x)^{-m}(1-\beta x)^{-m}(1-\gamma x)^{-m}\ldots.$

and $\qquad u^m = (1+u-1)^m$

$$= (1+u)^m - m\,(1+u)^{m-1} + \frac{m\,(m-1)}{2!}(1+u)^{m-2} - \ldots + (-)^m\,1.$$

Now, the coefficient of $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)\,x^n$ in

$$(1+u)^s = (1-\alpha x)^{-s}(1-\beta x)^{-s}(1-\gamma x)^{-s}\ldots$$

is

$$\left\{\frac{(s+p_1-1)!}{p_1!\,(s-1)!}\right\}^{\pi_1}\left\{\frac{(s+p_2-1)!}{p_2!\,(s-1)!}\right\}^{\pi_2}\left\{\frac{(s+p_3-1)!}{p_3!\,(s-1)!}\right\}^{\pi_3}\ldots.$$

Hence $\qquad A_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),\,(1^m)}$

$$= \left\{\frac{(m+p_1-1)!}{p_1!\,(m-1)!}\right\}^{\pi_1}\left\{\frac{(m+p_2-1)!}{p_2!\,(m-1)!}\right\}^{\pi_2}\left\{\frac{(m+p_3-1)!}{p_3!\,(m-1)!}\right\}^{\pi_3}\ldots$$

$$-m\left\{\frac{(m+p_1-2)!}{p_1!\,(m-2)!}\right\}^{\pi_1}\left\{\frac{(m+p_2-2)!}{p_2!\,(m-2)!}\right\}^{\pi_2}\left\{\frac{(m+p_3-2)!}{p_3!\,(m-2)!}\right\}^{\pi_3}\ldots$$

$$+\frac{m\,(m-1)}{2!}\left\{\frac{(m+p_1-3)!}{p_1!\,(m-3)!}\right\}^{\pi_1}\left\{\frac{(m+p_2-3)!}{p_2!\,(m-3)!}\right\}^{\pi_2}\left\{\frac{(m+p_3-3)!}{p_3!\,(m-3)!}\right\}^{\pi_3}\ldots$$

$-\ldots$ to $m+1$ terms.

Observe that, when

$$p_1 = p_2 = \ldots = \pi_1 = \pi_2 = \ldots = 1,$$

this expression reduces to the m^{th} divided difference of 0^n.

SECTION 5.

Parcels of species (m).

We now discuss what is commonly known as the distribution of n objects into m indifferent parcels, but here the objects are of type

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots).$$

We may separate the distribution function into portions corresponding to every partition of the number n into exactly m parts. First, consider such a partition which consists wholly of *unrepeated* parts, say $(r_1 r_2 r_3 \ldots r_m)$, $[\Sigma r = n]$,

the corresponding distribution function is necessarily

$$h_{r_1} h_{r_2} h_{r_3} \ldots h_{r_m},$$

but in any other case of distribution the function is much less simple.

For clearness first take $n = 4$, $m = 2$, and let us examine the distribution function corresponding to two objects in each parcel.

We have $h_2^2 = (\alpha^2+\beta^2+\gamma^2+\ldots+\alpha\beta+\beta\gamma+\gamma\alpha+\ldots)^2$,

and here the distribution, $\alpha\alpha$ in one parcel, $\beta\beta$ in the other, occurs twice instead of once, as would have to be the case if this were really the distribution function.

Take the expression

$$\frac{1}{(1-\alpha^2x^2)(1-\beta^2x^2)(1-\gamma^2x^2)\ldots(1-\alpha\beta x^2)(1-\beta\gamma x^2)(1-\gamma\alpha x^2)\ldots},$$

and expand it in ascending powers of x; herein the coefficient of x^{2s} will be the sum of order s of the homogeneous products of the quantities

$$\alpha^2, \beta^2, \gamma^2, \ldots \alpha\beta, \beta\gamma, \gamma\alpha, \ldots,$$

which compose the function h_2. This homogeneous product sum consists of a number of terms each of which is obtained by multiplying together s of the quantities

$$\alpha^2, \beta^2, \gamma^2, \ldots \alpha\beta, \beta\gamma, \gamma\alpha, \ldots,$$

repeated or unrepeated; clearly then, in this homogeneous product sum, the symmetric function

$$(p_1 p_2 p_3 \dots), \qquad [\Sigma p = 2s],$$

will occur just as many times as it is possible to distribute $2s$ objects $(p_1 p_2 p_3 \dots)$ into parcels (s), two objects being in each parcel.

If then we write

$$\frac{1}{(1-\alpha^2x^2)(1-\beta^2x^2)(1-\gamma^2x^2)\dots(1-\alpha\beta x^2)(1-\beta\gamma x^2)(1-\gamma\alpha x^2)\dots}$$
$$= 1 + h_{2^1}x^2 + h_{2^2}x^4 + h_{2^3}x^6 + \dots,$$

h_{2^s} will be the distribution function corresponding to the particular case of $2s$ objects in parcels (s), each parcel containing 2 objects.

Now take rs objects in parcels (s), each parcel containing r objects. Form a fraction whose denominator contains a factor corresponding to each component member of h_r, and then suppose

$$\frac{1}{\left[\begin{array}{l}(1-\alpha^r x^r)(1-\beta^r x^r)\dots(1-\alpha^{r-1}\beta x^r)(1-\alpha\beta^{r-1}x) \\ \qquad \dots(1-\alpha^{r-2}\beta^2 x^r)\dots(1-\alpha\beta\gamma\dots x^r)\end{array}\right]}$$
$$= 1 + h_{r^1}x^r + h_{r^2}x^{2r} + h_{r^3}x^{3r} + \dots.$$

Previous reasoning shows that the distribution function is

$$h_{r^s}.$$

Reserving for the present the particular examination of this important symmetric function, I continue the general discussion.

We have already considered the distribution function corresponding to the particular case of the partition of n into unrepeated parts; we are now in a position to determine the function corresponding to the case of r_1 parcels each containing t_1 objects, r_2 parcels each containing t_2 objects, &c., or say, corresponding to the partition of n,

$$(t_1^{r_1} t_2^{r_2} t_3^{r_3} \dots t_\nu^{r_\nu}) \qquad [\text{where } \Sigma r = m].$$

For, form the symmetric function

$$h_{t_1^{r_1}} h_{t_2^{r_2}} h_{t_3^{r_3}} \dots h_{t_\nu^{r_\nu}},$$

and observe the meaning of the coefficient of the symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots),$$

which will appear when the symmetric function product is developed.

The function $h_{\tau_1 t_1}$ contains terms corresponding to every selection of $t_1\tau_1$ objects of the total number n, and corresponding to every distribution of each of these selections into parcels (τ_1), each parcel containing exactly t_1 objects. Hence in the product

$$h_{t_1}^{\tau_1} h_{t_2}^{\tau_2} h_{t_3}^{\tau_3} \dots h_{t_\nu}^{\tau_\nu},$$

the symmetric function $\quad (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$

will occur just so many times as it is possible to distribute objects $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$ into m parcels, of which τ_1 contain exactly t_1 objects, τ_2 exactly t_2, τ_3 exactly t_3, &c., and τ_ν parcels exactly t_ν objects, &c.

Hence the particular distribution function sought for is

$$h_{t_1}^{\tau_1} \dots h_{t_\nu}^{\tau_\nu} \dots .$$

Finally, noticing that h_{r_1} and $h_{r'_1}$ are identical, we see that the distribution function of n objects into parcels (m) is

$$\Sigma\, h_{t_1}^{\tau_1} h_{t_2}^{\tau_2} h_{t_3}^{\tau_3} \dots,$$

the summation taking place over every partition

$$(t_1^{\tau_1} t_2^{\tau_2} t_3^{\tau_3} \dots)$$

of n which contains exactly $m\ [= \Sigma\tau]$ parts.

We may write this theorem in the form—

$$Dp\,(n), (m) = \Sigma A_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots),\ (m)} (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$$

$$= \Sigma\, h_{t_1}^{\tau_1} h_{t_2}^{\tau_2} h_{t_3}^{\tau_3} \dots,$$

where $\quad \Sigma\tau = m, \quad \Sigma\tau t = n.$

It is now clear that $Df\,(n)\,(m)$ is the coefficient of $x^n a^m$ in the expression

$$(1 + h_{1^1} xa + h_{1^2} x^2a^2 + \dots)(1 + h_{2^1} x^2a + h_{2^2} x^4a^2 + \dots)(1 + h_{3^1} x^3a + h_{3^2} x^6a^2 + \dots)\dots$$

$$\dots (1 + h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \dots)\dots$$

$$\equiv \prod_{s=1}^{s=\infty} (1 + h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \dots),$$

which is, therefore, its generating function.

Section 6.

Parcels of type $(m_1 m_2)$.

In this case, we are concerned with m_1 similar parcels of one kind and m_2 similar parcels of another kind. Of the n objects we may have m_1 objects only distributed amongst the m_1 similar parcels, and the remaining $n-m_1$ objects distributed amongst the m_2 similar parcels; or we may have m_1+s objects distributed amongst the m_1 similar parcels, and the remaining $n-m_1-s$ objects amongst the m_2 similar parcels, where $s \not> n-m_1-m_2$.

Hence

$$Dp\,(n),(m_1 m_2) = \sum_{s=0}^{n-m_1-m_2} Dp\,(m_1+s),(m_1)\,.\,Dp\,(n-m_1-s),(m_2),$$

and $Dp\,(n),(m_1m_2)$ is the coefficient of $x^n a^{m_1} b^{m_2}$ in the product

$$\prod_{s=1}^{s=\infty} (1+h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \ldots)(1+h_{s^1} x^s b + h_{s^2} x^{2s} b^2 + \ldots),$$

which is its generating function.

Section 7.

Parcels of type $(m_1m_2m_3\ldots)$.

By similar reasoning we find:—

$$Dp\,(n),(m_1m_2m_3) = \sum_{s=0}^{s=n-m_1-m_2-m_3} Dp(m_1+s),(m_1).Dp\,(n-m_1-s),(m_2m_3)$$

$$= \sum_{s=0}^{s=n-m_1-m_2-m_3} \Big[Dp\,(m_1+s),\,(m_1) \sum_{t=0}^{t=n-m_1-m_2-m_3-s} Dp\,(m_2+t),(m_2)\,.\,Dp\,(n-m_1-m_2-s-t),(m_3) \Big]$$

$$= \sum_{s=0}^{s=n-\Sigma m} \sum_{t=0}^{t=n-\Sigma m-s} Dp\,(m_1+s),(m_1)\,.\,Dp\,(m_2+t),(m_2) \times Dp\,(n+m_3-\Sigma m-s-t),\,(m_3),$$

and generally,

$$Dp(n),\,(m_1m_2m_3\ldots m_r) = \sum_{s_1=0}^{s_1=n-\Sigma m,} \sum_{s_2=0}^{s_2=n-\Sigma m-s,} \sum_{s_3=0}^{s_3=n-\Sigma m-s_1-s_2} \ldots$$

$$Dp\,(m_1+s_1),(m_1)\,.\,Dp\,(m_2+s_2),(m_2)\ldots Dp\,(n+m_r-\Sigma m-\Sigma s),(m_r)\,;$$

which is the coefficient of $x^n \mu_1^{m_1} \mu_2^{m_2} \ldots \mu_r^{m_r}$ in the product

$$\prod_{s=1}^{s=\infty} \prod_{t=1}^{t=\infty} (1+h_{s^1} x^s \mu_t + h_{s^2} x^{2s} \mu_t^2 + h_{s^3} x^{3s} \mu_t^3 + \ldots),$$

the generating function.

This determination completes analytically the solution of the problem of the distribution of objects $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ into parcels $(m_1 m_2 \ldots m_r)$.

Before proceeding to the subject of distributions, involving restrictions, I will draw up a list of some of the simpler results.

SECTION 8.

The simplest cases of Distribution into Parcels.

No. of Objects.	No. of Parcels.	Type of Parcels.	Distribution Function.
1	1	(1)	h_1,
2	1	(1)	h_2,
2	2	(2)	h_2,
2	2	(1^2)	h_1^2,
3	1	(1)	h_3,
3	2	(2)	h_2h_1,
3	2	(1^2)	$2h_2h_1$,
3	3	(3)	h_3,
3	3	(21)	h_2h_1,
3	3	(1^3)	h_1^3,
4	1	(1)	h_4,
4	2	(2)	$h_4+h_2^2$,
4	2	(1^2)	$2h_3h_1+h_2^2$,
4	3	(3)	h_2^2,
4	3	(21)	$h_2^2+h_2h_1^2$,
4	3	(1^3)	$3h_2h_1^2$,
4	4	(4)	h_4,
4	4	(31)	h_3h_1,
4	4	(2^2)	h_2^2,
4	4	(21^2)	$h_2h_1^2$,
4	4	(1^4)	h_1^4,
5	1	(1)	h_5,
5	2	(2)	$h_4h_1+h_3h_2$,
5	2	(1^2)	$2h_4h_1+2h_3h_2$,
5	3	(3)	$h_4h_1+h_3h_2-h_3h_1^2+h_2^2h_1$,
5	3	(21)	$h_4h_1+h_3h_2+2h_2^2h_1$,
5	3	(1^3)	$3h_3h_1^2+3h_2^2h_1$,

No. of Objects.	No. of Parcels.	Type of Parcels.	Distribution Function.
5	4	(4)	h_3h_2,
5	4	(31)	$h_3h_2+h^2h_1$,
5	4	(2^2)	$2h_2^2h_1$,
5	4	(21^2)	$2h_2^2h_1+h_2h_1^3$,
5	4	(1^4)	$4h_2h_1^3$,
6	1	(1)	h_6,
6	2	(2)	$h_6+2h_4h_2$,
6	2	(1^2)	$2h_5h_1+2h_4h_2+h_3^2$,
6	3	(3)	$h_6-h_5h_1+h_4h_2+h_4h_1^2+h_3^2-h_3h_2h_1+h_2^3$,
6	3	(21)	$2h_4h_2+h_4h_1^2+2h_3h_2h_1+h_2^3$,
6	3	(1^3)	$3h_4h_1^2+6h_3h_2h_1+h_2^3$,
6	4	(4)	$h_4h_2+h_3^2-h_3h_2h_1+h_2^3$,
6	4	(31)	$h_4h_1^2+h_3^2+h_3h_2h_1-h_3h_1^3+h_2^3+h_2^2h_1^2$,
6	4	(2^2)	$2h_4h_2+2h_2^3+h_2^2h_1^2$,
6	4	(21^2)	$h_4h_1^2+2h_3h_2h_1+h_2^3+3h_2^2h_1^2$,
6	4	(1^4)	$4h_3h_1^3+6h_2^2h_1^2$,
6	5	(5)	h_4h_2,
6	5	(41)	$h_4h_2+h_3h_2h_1$,
6	5	(32)	$h_3h_2h_1+h_2^3$,
6	5	(31^2)	$2h_3h_2h_1+h_2^2h_1^2$,
6	5	(21^2)	$h_2^3+2h_2^2h_1^2$,
6	5	(21^3)	$3h_2^2h_1^2+h_2h_1^4$,
6	5	(1^5)	$5h_2h_1^4$.

This table may be continued with little labour, the distribution functions being derived from those corresponding to a lesser number of objects whenever the parcel is of such a type that its partition contains more than a single part. For instance, we may employ either of the two formulæ

$$Dp\,(6), (31^2) = Dp\,(3), (3)\,Dp\,(3), (1^2) + Dp\,(4), (3)\,Dp\,(2), (1^2),$$

$$Dp\,(6), (31^2) = Dp\,(4), (31)\,Dp\,(2), (1) + Dp\,(5), (31)\,Dp\,(1), (1),$$

for the calculation of $Dp\,(6), (31^2)$.

The Distribution Functions can then be evaluated in terms of single partition forms by means of the tables subsequently given.

I proceed now to show how to express the symmetric function

$$h_{r^s}$$

in terms of $h_1, h_2, h_3, \ldots$ so as to obtain the expression generally of $Df(n)(m)$.

Section 9.

The symmetric function h_{r^s}.

This function is a homogeneous product sum, formed by taking s and s together the terms which compose the homogeneous product sum h_r. h_r is the homogeneous product sum of the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}.$$

Form the equation whose roots are the several terms of h_r, viz.,

$$x^p - j_1 x^{p-1} + j_2 x^{p-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

where $$p = \frac{(n+r-1)!}{(n-1)!\,r!}, \quad \text{and} \quad j_1 = h_{r^1} = h_r.$$

Form also the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

Let partitions in ( ) and [ ] denote respectively the symmetric functions of the roots of (i.) and (iii.), and σ_κ the sum of the κ^{th} powers of the roots of (ii.).

We may easily establish the two results

$$(\kappa) = (-)^{\kappa+1}[\kappa],$$
$$[\kappa^r] = (-)^{r(\kappa+1)}\sigma_\kappa\,;^*$$

* We have in fact

$$\frac{1}{1-\frac{a_1}{x}+\frac{a_2}{x^2}-\ldots+(-)^n\frac{a_n}{x^n}} = 1+\frac{h_1}{x}+\frac{h_2}{x^2}+\ldots+\frac{h_n}{x^n}+\ldots,$$

which may be written

$$\frac{1-\frac{\phi}{x^{n+1}}-\frac{\psi}{x^{n+2}}-\ldots}{1-\frac{a_1}{x}+\frac{a_2}{x^2}-\ldots+(-)^n\frac{a_n}{x^n}} = 1+\frac{h_1}{x}+\frac{h_2}{x^2}+\ldots+\frac{h_n}{x^n},$$

or

$$\frac{1-\frac{\phi}{x^{n+1}}-\frac{\psi}{x^{n+2}}-\ldots}{\left(1-\frac{\alpha}{x}\right)\left(1-\frac{\beta}{x}\right)\ldots} = \left(1+\frac{\alpha'}{x}\right)\left(1+\frac{\beta'}{x}\right)\ldots,$$

wherein $\alpha, \beta, \ldots$ are the roots of (i.), and $\alpha', \beta', \ldots$ the roots of (iii.).

whence $h_{r^1} = j_1 = \sigma_1 = [1^r]$,

$$h_{r^2} = j_1^2 - j_2 = \tfrac{1}{2}(\sigma + \sigma_2)$$
$$= \frac{\sigma_1^2}{2!} + \frac{\sigma_2}{2}$$
$$= \frac{[1^r]^2}{2!} + (-)^r \frac{[2^r]}{2};$$

Hence $$\log\left(1 - \frac{\phi}{x^{n+1}} - \frac{\psi}{x^{n+2}} - \dots\right) + \frac{(1)}{x} + \frac{1}{2}\frac{(2)}{x^2} + \dots + \frac{1}{n}\frac{(n)}{x^n} + \dots$$
$$= \frac{[1]}{x} - \frac{1}{2}\frac{[2]}{x^2} + \frac{1}{3}\frac{[3]}{x^3} - \dots + (-)^{n+1}\frac{[n]}{x^n} + \dots,$$

leading to the result

$$(\kappa) = (-)^{\kappa+1}[\kappa], \quad \text{where } \kappa \not> n.$$

Next, consider the identity

$$\frac{1}{\left(1-\frac{\alpha}{x}\right)\left(1-\frac{\beta}{x}\right)\left(1-\frac{\gamma}{x}\right)\dots} = \left(1+\frac{u}{x}\right)\left(1+\frac{v}{x}\right)\left(1+\frac{w}{x}\right)\dots,$$

wherein $u, v, w, \dots$ are the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \dots = 0,$$

and n is supposed indefinitely great.

Let the κ, κ^{th} roots of unity be denoted by

$$\epsilon_1, \epsilon_2, \epsilon_3, \dots \epsilon_\kappa;$$

then $$\prod_{s=1}^{s=\kappa} \frac{1}{\left(1-\frac{\epsilon_s\alpha}{x}\right)\left(1-\frac{\epsilon_s\beta}{x}\right)\left(1-\frac{\epsilon_s\gamma}{x}\right)\dots} = \prod_{s=1}^{s=\kappa}\left(1+\frac{\epsilon_s u}{x}\right)\left(1+\frac{\epsilon_s v}{x}\right)\left(1+\frac{\epsilon_s w}{x}\right)\dots,$$

or $$\frac{1}{\left(1-\frac{\alpha^\kappa}{x^\kappa}\right)\left(1-\frac{\beta^\kappa}{x^\kappa}\right)\left(1-\frac{\gamma^\kappa}{x^\kappa}\right)\dots}$$
$$= \left\{1+(-)^{\kappa+1}\frac{u^\kappa}{x^\kappa}\right\}\left\{1+(-)^{\kappa+1}\frac{v^\kappa}{x^\kappa}\right\}\left\{1+(-)^{\kappa+1}\frac{w^\kappa}{x^\kappa}\right\}\dots,$$

which is

$$\frac{1}{1-\frac{(\kappa)}{x^\kappa}+\frac{(\kappa^2)}{x^{2\kappa}}-\frac{(\kappa^3)}{x^{3\kappa}}+\dots} = 1+(-)^{\kappa+1}\frac{[\kappa]}{x^\kappa}+(-)^{2(\kappa+1)}\frac{[\kappa^2]}{x^{2\kappa}}+\dots+(-)^{r(\kappa+1)}\frac{[\kappa^r]}{x^{r\kappa}}+\dots.$$

The coefficient of $\frac{1}{x^{r\kappa}}$ in the development of the sinister of this identity, according to ascending powers of $\frac{1}{x^\kappa}$, is the homogeneous product sum of order r of the quantities $\alpha^\kappa, \beta^\kappa, \gamma^\kappa, \dots$; it is thus equal to σ_κ, the sum of the κ^{th} powers of the roots of (ii.).

Hence $$\sigma_\kappa = (-)^{r(\kappa+1)}[\kappa^r],$$
which is equivalent to the second of the two results.

$$h_{r^3} = j_1^3 - 2j_1j_2 + j_3$$

$$= \frac{\sigma_1^3}{3!} + \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3}$$

$$= \frac{[1^r]^3}{3!} + (-)^r \frac{[1^r][2^r]}{2} + \frac{[3^r]}{3};$$

$$h_{r^4} = j_1^4 - 3j_1^2 j_2 + j_2^2 + 2j_1j_3 - j_4$$

$$= \frac{\sigma_1^4}{4!} + \frac{\sigma_1^2\sigma_2}{2!\,2} + \frac{\sigma_1\sigma_3}{3} + \frac{\sigma_2^2}{2^2.\,2!} + \frac{\sigma_4}{4}$$

$$= \frac{[1^r]^4}{4!} + (-)^r \frac{[1^r]^2[2^r]}{2!\,2} + \frac{[1^r][3^r]}{3} + \frac{[2^r]^2}{2^2.\,2!} + (-)^r \frac{[4^r]}{4},$$

and so forth.

The law is identical with that which obtains in the expression of the elementary symmetric functions in terms of the sums of powers, with the exception that the signs are all positive when r is even.

Hence we can express h_{rs} in terms of $h_1, h_2, h_3, \ldots$.

In particular we thus find

$$h_{1^1} = h_1,$$

and generally

$$h_{1^r} = h_{r^1} = h_r,$$

$$h_{2^2} = \frac{1}{2!}\{h_2^2 + h_2^2 - 2h_1h_3 + 2h_4\} = h_2^2 - h_1h_3 + h_4,$$

$$h_{2^3} = \frac{1}{3!}\{h_2^3 + 3h_2(h_2^2 - 2h_1h_3 + 2h_4)$$
$$+ 2(h_2^3 - 3h_3h_2h_1 + 3h_3^2 + 3h_4h_1^2 - 3h_4h_2 - 3h_5h_1 + 3h_6)\}$$
$$= h_6 - h_5h_1 + h_4h_1^2 + h_3^2 - 2h_3h_2h_1 + h_2^3,$$

$$h_{3^2} = \frac{1}{2!}\{h_3^2 - h_3^2 + 2h_4h_2 - 2h_5h_1 + 2h_6\} = h_6 - h_5h_1 + h_4h_2,$$

$$h_{2^4} = \frac{1}{4!}\{h_2^4 + 6h_2^2(h_2^2 - 2h_1h_3 + 2h_4)$$
$$+ 8h_2(h_2^3 - 3h_3h_2h_1 + 3h_3^2 + 3h_4h_1^2 - 3h_4h_2 - 3h_5h_1 + 3h_6)$$
$$+ 3(h_2^4 + 4h_3^2h_1^2 + 4h_4^2 - 4h_3h_2^2h_1 + 4h_4h_2^2 - 8h_4h_3h_1)$$
$$+ 6(h_2^4 - 4h_3h_2^2h_1 + 2h_3^2h_1^2 + 4h_3^2h_2 + 4h_4h_2h_1^2 - 4h_4h_2^2$$
$$- 8h_4h_3h_1 + 6h_4^2 - 4h_5h_1^3 + 8h_5h_2h_1 - 4h_5h_3$$
$$+ 4h_6h_1^2 - 4h_6h_2 - 4h_7h_1 + 4h_8)\}$$
$$= h_2^4 - 3h_3h_2^2h_1 - h_4h_2^2 + 2h_3^2h_2 + 2h_4h_2h_1^2 + 2h_5h_2h_1 + h_3^2h_1^2 + 2h_4^2$$
$$- 3h_4h_3h_1 - h_5h_1^3 - h_5h_3 + h_6h_1^2 - h_6h_2 - h_7h_1 + h_8$$
$$= h_8 - h_7h_1 - h_6h_2 + h_6h_1^2 - h_5h_3 + 2h_5h_2h_1 - h_5h_1^3 + 2h_4^2 - 3h_4h_3h_1 - h_4h_2^2$$
$$+ 2h_4h_2h_1^2 + 2h_3^2h_2 + h_3^2h_1^2 - 3h_3h_2^2h_1 + h_2^4,$$

$$h_{4^2} = \frac{1}{2!}\{h_4^2+h_4^2-2h_5h_3+2h_6h_2-2h_7h_1+2h_8\}$$

$$= h_8-h_7h_1+h_6h_2-h_5h_3+h_4^2,$$

and for present purposes we need calculate no further.

Section 10.

Groups of type (1^m).

Consider the expansion of

$$h_1^n = (\alpha+\beta+\gamma+\ldots)^n.$$

It consists of products of the quantities $\alpha, \beta, \gamma, \ldots$ of the n^{th} degree taken in all possible ways, repetitions and permutations being alike allowable. On this understanding the expansion consists of a number of terms each with coefficient unity. Suppose any such term to be

$$\alpha_1\beta_1\beta_2\alpha_2\alpha_3\gamma_1\beta_3\alpha_4\ldots,$$

and place dots in any $m-1$ out of the $n-1$ intervals between the letters; this can be done in

$$\frac{(n-1)!}{(n-m)!\,(m-1)!}\text{ ways.}$$

A distribution (1^m) will correspond to each of these ways for every term of the expansion h_1^n.

Thus the distribution function of n objects into groups (1^m) is

$$Dg\,(n), (1^m) = \frac{(n-1)!}{(n-m)!\,(m-1)!}\,h_1^n,$$

and denoting by $$B_{(p_1^{\pi_1}p_2^{\pi_2}\ldots),\,(1^m)}$$

the number of distributions of objects $(p_1^{\pi_1}p_2^{\pi_2}\ldots)$ into groups (1^m), we have

$$B_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),\,(1^m)} = \frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}(p_3!)^{\pi_3}\ldots}\cdot\frac{(n-1)!}{(n-m)!\,(m-1)!}.$$

The distribution function is the coefficient of x^n in

$$(h_1x+h_1^2x^2+h_1^3x^3+h^4x^4+\ldots)^m.$$

Section 11.

Groups of type (m).

Consider the symmetric function sum

$$\Sigma\frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots}(p_1^{\pi_1}p_2^{\pi_2}\ldots) = h_1^n$$

arranged as a sum of products of letters $\alpha, \beta, \gamma, \ldots$, each permutation of every product occurring as a term, so that only coefficients equal to unity present themselves.

We require the homogeneous product sum of all these terms, of any desired order.

Putting $n = 2, 3, \ldots r$ successively, we may write as generating functions

$$\frac{1}{(1-\alpha^2ax^2)(1-\beta^2ax^2)\ldots(1-\alpha\beta ax^2)^2(1-\alpha\gamma ax^2)^2\ldots}$$
$$= 1+H_2ax^2+H_{2^2}a^2x^4+H_{2^3}a^3x^6+\ldots$$

$$\frac{1}{(1-\alpha^3ax^3)(1-\beta^3ax^3)\ldots(1-\alpha^2\beta ax^3)^3\ldots(1-\alpha\beta\gamma ax^3)^6\ldots}$$
$$= 1+H_3ax^3+H_{3^2}a^2x^6+\ldots,$$

&c.,

and generally

$$\frac{1}{(1-\alpha^rax^r)\ldots(1-\alpha^{r-1}\beta ax^r)^r\ldots(1-\alpha\beta\gamma\ldots ax^r)^{r!}\ldots}$$
$$= 1+H_rax^r+H_{r^2}a^2x^{2r}+H_{r^3}a^3x^{3r}+\ldots,$$

wherein H_{r^s} represents the s^{th} order homogeneous product sum of all the separate terms which arise when h_1^r is multiplied out *in extenso.*

By reasoning similar to that employed in the discussion of "Parcels," we see that

$$H_{r^s}$$

denotes the distribution function of sr objects in groups (s) in suchwise that each group shall consist of r objects.

Also that the distribution function of n objects into groups (m) is

$$\Sigma H_{t_1^{r_1}} H_{t_2^{r_2}} H_{t_3^{r_3}} \ldots,$$

the summation taking place over every partition of n

$$(t_1^{r_1}\, t_2^{r_2}\, t_3^{r_3} \ldots)$$

which contains exactly $m\ [=\Sigma r]$ parts.

Thus $$Dg\,(n), (m) = \Sigma H_{t_1^{r_1}} H_{t_2^{r_2}} H_{t_3^{r_3}} \ldots,$$

and it is the coefficient of $x^n a^m$ in the expansion of

$$\prod_{s=1}^{s=\infty} (1+H_s x^s a + H_{s^2} x^{2s} a^2 + \ldots),$$

which is, therefore, the generating function.

Section 12.

Groups of type $(m_1 m_2 m_3 \ldots)$.

The law of derivation of the distribution functions of groups of many part partition types is precisely the same in the case of groups as in the case of parcels.

I, therefore, proceed at once to the examination of the new symmetric function

$$H_{r^s}.$$

Section 13.

The symmetric function H_{r^s}.

Let

$$x^w - k_1 x^{w-1} + k_2 x^{w-2} - \ldots = 0$$

be the equation, having for its roots the several quantities of which H_{r^s} is the homogeneous product sum of order s.

Then

$$k_1 = h_1^r = H_{r^1}.$$

Further, let σ_t denote the sum of the t^{th} powers of the roots of this equation.

If partitions in ( ) refer to the symmetric functions of the equation

$$(x-\alpha)(x-\beta)(x-\gamma) \ldots = 0,$$

we have

$$\sigma_t = (t)^r;$$

also

$$k_2 = \tfrac{1}{2}(\sigma_1^2 - \sigma_2) = \frac{(1)^{2r}}{2\,!} - \frac{(2)^r}{2};$$

hence

$$H_{r^2} = k_1^2 - k_2 = \frac{\sigma_1^2}{2!} + \frac{\sigma_2}{2}$$

$$= \frac{(1)^{2r}}{2\,!} + \frac{(2)^r}{2}.$$

Also, since

$$k_3 = \frac{\sigma_1^3}{3!} - \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3},$$

we find

$$H_{r^3} = k_1^3 - 2k_1k_2 + k_3$$

$$= \frac{\sigma_1^3}{3\,!} + \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3}$$

$$= \frac{(1)^{3r}}{3\,!} + \frac{(1)^r(2)^r}{2} + \frac{(3)^r}{3},$$

and so forth.

Hence, finally, transforming as before to symmetric functions of the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0,$$

$$H_{r^1} = (1)^r = [1]^r,$$

$$H_{r^2} = \frac{[1]^{2r}}{2\,!} + (-)^r \frac{[2]^r}{2},$$

$$H_{r^3} = \frac{[1]^{3r}}{3\,!} + (-)^r \frac{[1]^r [2]^r}{2} + \frac{[3]^r}{3},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

The symmetric function H_{r^s} can be thus expressed in terms of h_1, h_2, h_3.

These results should be compared with those obtained in section 9 for the case of distribution into parcels.

It will be noticed that H_{r^s} is derived from h_{r^s} by simply writing $[k]^r$ in place of $[k^r]$.

Section 14.

Restricted distributions into Parcels.

The distributions considered in the foregoing sections were not subject to any restriction. There was no limit to the number of similar objects that it was permissible to distribute either into a single parcel or into a set of similar parcels. This freedom from restriction led naturally to the invariable appearance of the symmetric functions, which express the sums of the homogeneous products of the quantities, in the distribution functions.

In order to find the distribution function of n objects in n parcels, one object in each parcel, subject to the restriction that no two objects of the same kind are to appear in parcels of the same kind, we have merely to employ the elementary symmetric functions

$$a_1, a_2, a_3, \ldots$$

instead of the homogeneous product sums

$$h_1, h_2, h_3, \ldots.$$

The product $\qquad a_{p_1} a_{q_1} a_{r_1} \ldots$

is necessarily the distribution function of n objects into parcels $(p_1 q_1 r_1 \ldots)$, where $p_1 + q_1 + r_1 + \ldots = n$, subject to the restriction that no two similar objects are to appear in similar parcels. Thus, since

$$a_3 a_2 = (1^3)(1^2) = (2^2 1) + 3(21^3) + 10(1^5),$$

we discover that, subject to the restriction, objects (21^3) can be distributed into parcels (32) in three different ways. These three ways are apparent in the scheme:—

A	A	A	B	B
α	β	γ	δ	α
α	γ	δ	β	α
α	δ	β	γ	α

We wish now to impose the restriction that not more than t similar objects are to be distributed into similar parcels. For this purpose, form the symmetric functions

$$t_1, t_2, t_3, t_4, \dots,$$

where t_s is defined to be that portion of the homogeneous product sum h_s in which no quantity occurs to a higher power than t.

In the product $$t_{p_1} t_{q_1} t_{r_1} \dots,$$

we may suppose any term composing t_{p_1} to be written out with the letters in any order and a dot placed between each consecutive pair of letters. We consider the p_1 letters to denote p_1 objects distributed into p_1 similar parcels. Obviously, not more than t similar objects thus appear in similar parcels. By reasoning similar to that employed in section 1, it is established that, in the product

$$t_{p_1} t_{q_1} t_{r_1} \dots,$$

when expanded, the symmetric function $(pqr\dots)$ will appear with a coefficient which represents the number of ways that it is possible to distribute objects $(pqr\dots)$ into parcels $(p_1 q_1 r_1 \dots)$, one object in each parcel, subject to the restriction that not more than t similar objects are to appear in similar parcels. This restriction does not alter the reciprocal nature of the distribution. It is immaterial whether we regard the objects distributed into the parcels or the parcels distributed into the objects. We may say that not more than t similar objects are to be contained in similar parcels, or we may say that not more than t similar parcels are to contain similar objects. The restriction does not affect the reciprocity.

Theorem.—The number of ways of distributing objects $(pqr\dots)$ into parcels $(p_1 q_1 r_1 \dots)$ is equal to the number of ways of distributing objects $(p_1 q_1 r_1 \dots)$ into parcels $(pqr\dots)$; the distributions being subject to the restriction that not more than t similar objects are to present themselves in similar parcels.

This theorem points to a general algebraic law of symmetry.

Theorem.—The coefficient of symmetric function $(pqr\ldots)$ in the development of

$$t_{p_1} t_{q_1} t_{r_1} \ldots$$

is equal to the coefficient of symmetric function $(p_1 q_1 r_1 \ldots)$ in the development of

$$t_p t_q t_r \ldots .$$

This theorem includes all previous laws of symmetry.

The observation is made that, if any table of functions be found to possess symmetry of this nature, it follows, as a necessary and easily established result, that the "inverse table" also possesses the same symmetry.

The laws of symmetry, as apparent in ordinary tables of symmetric functions, are included in the above theorem. Still retaining the same restriction, it is easy to prove that the distribution function of n objects into parcels (1^m) is

$$\Sigma\, {}_tA_{(pqr\ldots),\,(1^m)}\,(pqr\ldots) = \Sigma \frac{m\,!}{\pi_1\,!\;\pi_2\,!\;\pi_3\,!\ldots}\, t_{p_1}^{\pi_1} t_{p_2}^{\pi_2} t_{p_3}^{\pi_3} \ldots,$$

wherein $\Sigma\pi = m;\quad \Sigma p\pi = n.$

SECTION 15.

General value of ${}_tA_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3}\ldots),\,(1^m)}$.

We require the coefficient of x^n in the expansion of

$$(t_1x + t_2x^2 + t_3x^3 + \ldots)^m,$$

and therein, the coefficient of the symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3}\ldots).$$

Put $\quad t_1x + t_2x^2 + t_3x^3 + \ldots = u,$

so that $\quad 1+u = \frac{1-\alpha^{t+1}x^{t+1}}{1-\alpha x}\cdot\frac{1-\beta^{t+1}x^{t+1}}{1-\beta x}\cdot\frac{1-\gamma^{t+1}x^{t+1}}{1-\gamma x}\ldots,$

and
$$(1+u)^m = \Pi\left\{1 - m\alpha^t x^t + \frac{m(m-1)}{2!}\alpha^{2t}x^{2t} - \ldots\right\}$$
$$\times\left\{1 + m\alpha x + \frac{m(m+1)}{2!}\alpha^2x^2 + \ldots\right\}.$$

In this product, the coefficient of the symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3}\ldots)$$

is
$$\left\{\frac{(m+p_1-1)!}{(m-1)!\,p_1!}-m\frac{(m+p_1-t-2)!}{(m-1)!\,(p_1-t-1)!}+\dots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-1)!}{(m-1)!\,p_2!}-m\frac{(m+p_2-t-2)!}{(m-1)!\,(p_2-t-1)!}+\dots\right\}^{\pi_2}\dots,$$

and, since
$$u^m=(1+u-1)^m=(1+u)^m-m(1+u)^{m-1}+\frac{m(m-1)}{2!}(1+u)^{m-2}-\dots,$$

we find
$${}_tA_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots),\,(1^m)}$$
$$=\left\{\frac{(m+p_1-1)!}{(m-1)!\,p_1!}-m\frac{(m+p_1-t-2)!}{(m-1)!\,(p_1-t-1)!}+\dots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-1)!}{(m-1)!\,p_2!}-m\frac{(m+p_2-t-2)!}{(m-1)!\,(p_2-t-1)!}+\dots\right\}^{\pi_2}\dots$$
$$-m\left\{\frac{(m+p_1-2)!}{(m-2)!\,p_1!}-(m-1)\frac{(m+p_1-t-3)!}{(m-2)!\,(p_1-t-1)!}+\dots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-2)!}{(m-2)!\,p_2!}-(m-1)\frac{(m+p_2-t-3)!}{(m-2)!\,(p_2-t-1)!}+\dots\right\}^{\pi_2}\dots$$
$$+\frac{m(m-1)}{2!}\left\{\frac{(m+p_1-3)!}{(m-3)!\,p_1!}-(m-2)\frac{(m+p_1-t-4)!}{(m-3)!\,(p_1-t-1)!}+\dots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-3)!}{(m-3)!\,p_2!}-(m-2)\frac{(m+p_2-t-4)!}{(m-3)!\,(p_2-t-1)!}+\dots\right\}^{\pi_2}\dots$$
$$-\dots\dots.$$

There is no difficulty in continuing the theory of this restriction. I have not thought it advantageous to proceed further with it in the case of distributions into parcels.

Section 16.

Restricted Distributions into Groups.

It is convenient to write
$$h_1^s=H_s.$$

The distribution function of the unrestricted distribution of n objects into groups
$$(1^m)$$

is then the coefficient of x^n in the expansion of
$$(H_1x+H_2x^2+H_3x^3+\dots)^m,$$

where
$$H_1 = h_1 = (1),$$
$$H_2 = h_1^2 = (2)+2\,(1^2),$$
$$H_3 = h^3 = (3)+3\,(21)+6\,(1^3),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

I further denote by $A_s, B_s, C_s, \ldots T_s, \ldots$

those portions of H_s which involve partitions containing no part greater than

$$1, 2, 3, \ldots t, \ldots \text{ respectively.}$$

It is easily seen that the distribution function of n objects into groups (1^m), subject to the restriction that not more than t objects of the same kind are to present themselves in groups of the same kind, is given by the coefficient of x^n in

$$(T_1x+T_2x^2+T_3x^3+\ldots)^m.$$

Section 17.

Algebraic Theorems derived from the Theory of Distributions.

Definition.

Of a number n, take any partition

$$(\lambda_1\lambda_2\lambda_3 \ldots \lambda_s).$$

It becomes necessary to consider the separation of such a partition into component partitions. Such a separation may be represented by enclosing the component partitions in brackets; thus:

$$(\lambda_1\lambda_2)\,(\lambda_3\lambda_4\lambda_5)\,(\lambda_6) \ldots .$$

It is convenient to arrange the components in descending order as regards their weight or content, and, if these successive weights are in order

$$p, q, r, \ldots,$$

to speak of a separation of species $(pqr\ldots)$.

Just as we speak of the degree of a partition, meaning the magnitude of the largest part in such partition, so we may speak of the degree of a separation, meaning the sum of the largest parts in its components.

We have thus, primarily, three characteristics of a separation, viz.,

(i.) the separable partition,

(ii.) the species,

(iii.) the degree.

General Theorem of Algebraic Reciprocity.

In § 1, I considered the distribution of n objects into n parcels, and showed that the distribution function of objects into parcels

$$(p_1 q_1 r_1 \dots)$$

is

$$h_{p_1} h_{q_1} h_{r_1} \dots .$$

We may analyse this result in the following manner:—

$$\begin{aligned} \text{Write}\quad X_1 &= (1)\, x_1, \\ X_2 &= (2)\, x_2 + (1^2)\, x_1^2, \\ X_3 &= (3)\, x_3 + (21)\, x_2 x_1 + (1^3)\, x_1^3, \\ X_4 &= (4)\, x_4 + (31)\, x_3 x_1 + (2^2)\, x_2^2 + (21^2)\, x_2 x_1^2 + (1^4)\, x_1^4, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

and generally $\qquad X_s = \Sigma\, (\lambda\mu\nu\dots)\, x_\lambda x_\mu x_\nu \dots,$

the summation being in regard to every partition of s.

Consider the result of multiplication

$$X_{p_1} X_{q_1} X_{r_1} \dots = \Sigma P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \dots .$$

P consists of an aggregate of terms, each of which, to a numerical factor *près*, is a separation of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots)$$

of species

$$(p_1 q_1 r_1 \dots).$$

P, further, is the distribution function of objects into parcels

$$(p_1 q_1 r_1 \dots),$$

subject to certain restrictions.

If in any distribution of n objects into n parcels (one object into each parcel) we write down a number

$$\xi$$

whenever we observe ξ similar objects in similar parcels, we write down a succession of numbers

$$\xi_1, \xi_2, \xi_3, \dots,$$

where

$$(\xi_1 \xi_2 \xi_3 \dots)$$

is some partition of n.

We may be given these numbers, and say that the distribution is subject to a restriction of partition

$$(\xi_1 \xi_2 \xi_3 \dots).$$

Subject to this restriction, there are a certain number of distributions. In the present case, if we put

$$x_1 = x_2 = x_3 = \ldots = 1,$$

$$\Sigma P$$

is obviously the distribution function of n objects into n parcels without restriction.

P itself is manifestly the distribution function subject to the restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots).$$

Employing a more general notation, we may write

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \Sigma P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

and then P is the distribution function of objects into parcels

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

subject to the restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots).$$

Multiplying out P, we get the result

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \Sigma \theta \, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

indicating that, with a restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots),$$

there are precisely θ ways of distributing n objects

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots)$$

amongst n parcels $\quad (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$

one object into each parcel.

Now, it is seen intuitively that, since there is one object in every parcel, it is immaterial whether we regard an object attached to a parcel or a parcel attached to an object, and that making this exchange does not alter the partition of restriction.

Hence the number of distributions must be the same, and if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \ldots + \theta \, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

then also $\quad X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} X_{\lambda_3}^{l_3} \ldots = \ldots + \theta \, (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots.$

This extensive theorem of algebraic reciprocity includes all known theorems of symmetry in symmetric functions.

Limiting attention to the powers of

$$x_1,$$

we immediately obtain Cayley's law of symmetry.

Putting, further, $x_1 = 0,$

we obtain a theorem of wide application in the multiplication of covariants of binary quantics.

We may enunciate it as follows :—

Theorem.—Selecting at pleasure any three partitions of n

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots),$$

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots),$$

separate in any manner the numbers occurring in

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots)$$

into π_1 portions of content p_1,

π_2 „ „ p_2,

π_3 „ „ p_3,

$\vdots$ „ „ $\vdots$

Multiply the product of partitions thus formed by the number which expresses the number of ways of permuting the product, the only permutations allowable being those amongst partitions of the same content; take the sum of all such separations of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots),$$

each multiplied by the proper number determined as explained above. The coefficient of the symmetric function

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots),$$

in this sum of compound symmetric functions, will be precisely the same as if in the process we had interchanged the partitions

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots), \quad (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots).$$

Generalisation of Waring's Formula.

Waring's formula for the expression of the n^{th} power sum of the roots of an equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0,$$

in terms of the coefficients, is usually written

$$S_m = \Sigma \frac{(-)^{m+\Sigma\lambda}(\Sigma\lambda-1)!\,m}{\lambda_1!\,\lambda_2!\,\ldots\,\lambda_n!}\,a_1^{\lambda_1}a_2^{\lambda_2}\ldots a_n^{\lambda_n}.$$

Write this in the following form, viz.,

$$\frac{(-)^m(m-1)!}{m!}\,S(1^m) = \Sigma\frac{(-)^{\Sigma\lambda}(\Sigma\lambda-1)!}{\lambda_1!\,\lambda_2!\,\ldots\,\lambda_n!}\,(1)^{\lambda_1}(1^2)^{\lambda_2}\ldots(1^n)^{\lambda_n}.$$

Observe that this formula expresses the sum of the m^{th} powers of the roots in terms of separations of the partition

$$(1^m);$$

the typical separation $(1)^{\lambda_1}(1^2)^{\lambda_2}\ldots(1^n)^{\lambda_n}$

is of species $(1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n})$,

and of degree $\Sigma\lambda$.

I proceed to demonstrate a formula for the expression of the m^{th} power sum of the roots as a linear function of separations of any partition whatever of m.

The general formula to be established is

$$(-)^{l+m+\ldots}\frac{(l+m+\ldots-1)!}{l!\,m!\,\ldots}\,S(\lambda^l\mu^m\ldots)$$
$$= \Sigma\,(-)^{j_1+j_2+\ldots}\frac{(j_1+j_2+\ldots-1)!}{j_1!\,j_2!\,\ldots}\,(J_1)^{j_1}(J_2)^{j_2}\ldots,$$

wherein $(\lambda^l\mu^m\ldots)$ is the separable partition, $(J_1)^{j_1}(J_2)^{j_2}\ldots$ is a separation of $(\lambda^l\mu^m\ldots)$, and the summation is in regard to every such separation.

In this formula, $S(\lambda^l\mu^m\ldots)$

denotes the sum of the n^{th} powers of the roots $(l\lambda+m\mu+\ldots = n)$ in terms of separations of $(\lambda^l\mu^m\ldots)$.

Write down the series of relations

$$(a_1) = S_{a_1},$$
$$(a_1a_2) = S_{a_1}S_{a_2} - S_{a_1+a_2},$$
$$(a_1a_2a_3) = S_{a_1}S_{a_2}S_{a_3} - S_{a_1}S_{a_2+a_3} - S_{a_2}S_{a_1+a_3} - S_{a_3}S_{a_1+a_2} + 2S_{a_1+a_2+a_3},$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

from the first two of these we find

$$S_{a_1+a_2} = (a_1)(a_2) - (a_1a_2),$$

or, as this may be written,

$$S(a_1a_2) = (a_1)(a_2) - (a_1a_2),$$

and from the third we get, after reduction,

$$2S(a_1a_2a_3) = 2(a_1)(a_2)(a_3) - (a_1)(a_2a_3) - (a_2)(a_1a_3) - (a_3)(a_1a_2) + (a_1a_2a_3).$$

It is obvious that we can continue this series indefinitely, and express $S(a_1a_2a_3a_4)$, $S(a_1a_2a_3a_4a_5)$, ... in terms of separations of the partitions

$$(a_1a_2a_3a_4),\quad (a_1a_2a_3a_4a_5), \ldots .$$

This holds also notwithstanding any equalities that may exist between the parts $a_1, a_2, a_3, \ldots$ of the separable partition. The formulæ would, however, require modification in those cases.

First, suppose that no equalities exist between the parts of the separable partition; we require the expression of

$$S(a_1a_2 \ldots a_n)$$

in terms of separations of $\quad (a_1a_2 \ldots a_n)$.

One such separation is, for example,

$$(a_{11}a_{12} \ldots a_{1p})(a_{21}a_{22} \ldots a_{2q}) \ldots (a_{t1}a_{t2} \ldots a_{t\nu}),$$

where the successive component partitions have $p, q, \ldots \nu$ parts, respectively, and there are t components.

Of this type there are in all

$$\frac{n!}{p!\,q!\ldots\nu!} \text{ separations,}$$

and by symmetry we see that in the expression of

$$S(a_1a_2 \ldots a_n),$$

each such separation must be affected by the same coefficient.

Write, then,

$$S(a_1a_2 \ldots a_n) = \Sigma P \Sigma (a_{11}a_{12} \ldots a_{1p})(a_{21}a_{22} \ldots a_{2q}) \ldots (a_{t1}a_{t2} \ldots a_{t\nu}).$$

To determine P, observe that if

$$a_1 = a_2 = \ldots = a_n,$$

the formula should reduce to Waring's, viz.—

$$S(a_1^n) = \Sigma(-)^{n+t}(t-1)!\,n(a_1^p)(a_1^q) \ldots (a_1^\nu),$$

where for simplicity it is supposed that no equalities exist between the ntegers $p, q, \ldots \nu$.

On this supposition of the equality of the parts of the separable partition, the assumed formula becomes

$$S(a_1^n) = \Sigma P \frac{n\,!}{p\,!\,q\,!\,\dots\,\nu\,!}\, p\,!\,(a_1^p)\, q\,!\,(a_1^q)\,\dots\,\nu\,!\,(a_1^\nu),$$

and, equating these two expressions for $S(a_1^n)$, we find

$$P = (-)^{n+t} \frac{(t-1)\,!}{(n-1)\,!},$$

and we thus reach the formula

$$(-)^n (n-1)\,!\; S(a_1 a_2 \dots a_n)$$

$$= \Sigma\,(-)^t (t-1)\,!\; \Sigma\,(a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{t\nu}),$$

or, as this may be written,

$$(-)^n (n-1)\,!\; S(a_1 a_2 \dots a_n)$$

$$= \Sigma\,(-)^t (t-1)\,!\; (a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{t\nu}).$$

The supposition of any number of equalities between the integers $p, q, \dots \nu$ renders requisite an easy modification of the proof, and leads to the same final result.

I pass on to the general case

$$S(\lambda^l \mu^m \dots),$$

and put

$$S(\lambda^l \mu^m \dots) = \Sigma P\,(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}(\lambda^{l_2}\mu^{m_2}\dots)^{j_2}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu}.$$

Starting with the formula

$$(-)^n (n-1)\,!\; S(a_1 a_2 \dots a_n)$$

$$= \Sigma\,(-)^t (t-1)\,!\; (a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{tu}),$$

suppose that of the numbers

$$p, q, \dots u,$$

j_1 have the value $l_1 + m_1 + \dots$,

j_2 have the value $l_2 + m_2 + \dots$,

... ...

j_ν have the value $l_\nu + m_\nu + \dots$.

We may give such values to the quantities a, that certain of the separations under the summation sign shall become

$$(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}(\lambda^{l_2}\mu^{m_2}\dots)^{j_2}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu};$$

viz.,—we must put l of them equal to λ, m of them equal to μ, and so on. The number of separations which thus become of the required form is easily found to be

$$\frac{l\,!\,m\,!}{(l_1!\,m_1!\,\ldots)^{j_1}\,(l_2!\,m_2!\,\ldots)^{j_2}\ldots(l_\nu!\,m_\nu!\,\ldots)^{j_\nu}\,j_1!\,j_2!\ldots j_\nu!}.$$

Also, on replacing a component $(a_{11}\,a_{12}\ldots a_{1p})$ by $(\lambda^{l_1}\mu^{m_1}\ldots)$, where

$$l_1+m_1+\ldots=p,$$

we must multiply by $\quad l_1!\,m_1!\ldots;$

we thus get a multiplier

$$(l_1!\,m_1!\ldots)^{j_1}\,(l_2!\,m_2!\ldots)^{j_2}\ldots(l_\nu!\,m_\nu!\ldots)^{j_\nu},$$

and, further, t is equivalent to Σj.

Thus,

$$P=(-)^{l+m+\ldots+\Sigma j}\frac{(\Sigma j-1)!}{(l+m+\ldots-1,)!}$$

$$\times\frac{l!\,m!\ldots}{(l_1!\,m_1!\ldots)^{j_1}\,(l_2!\,m_2!\ldots)^{j_2}\ldots(l_\nu!\,m_\nu!\ldots)^{j_\nu}j_1!\,j_2!\ldots j_\nu!}$$

$$\times(l_1!\,m_1!\ldots)^{j_1}\,(l_2!\,m_2!\ldots)^{j_2}\ldots(l_\nu!\,m_\nu!\ldots)^{j_\nu}$$

$$=(-)^{l+m+\ldots+\Sigma j}\frac{l!\,m!\ldots}{(l+m+\ldots-1)!}\cdot\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots j_\nu!},$$

leading to the formula

$$\frac{(-)^{l+m+\ldots}(l+m+\ldots-1)!}{l!\,m!\ldots}S(\lambda^l\mu^m\ldots)$$

$$=\Sigma(-)^{\Sigma j}\frac{(\Sigma j-1)}{j_1!\,j_2!\ldots j_\nu!}(\lambda^{l_1}\mu^{m_1}\ldots)^{j_1}(\lambda^{l_2}\mu^{m_2}\ldots)^{j_2}\ldots(\lambda^{l_\nu}\mu^{m_\nu}\ldots)^{j_\nu}.$$

Assuming the form

$$S(\lambda^l\mu^m\ldots)=\Sigma P\Sigma(\lambda^{l_1}\mu^{m_1}\ldots)^{j_1}(\lambda^{l_2}\mu^{m_2}\ldots)^{j_2}\ldots(\lambda^{l_\nu}\mu^{m_\nu}\ldots)^{j_\nu},$$

another proof may be given.

In this form the sums—

$$l_1+m_1+\ldots,$$
$$l_2+m_2+\ldots,$$
$$\ldots\quad\ldots\quad\ldots$$
$$l_\nu+m_\nu+\ldots,$$

are each considered constant.

Putting each part equal to λ, we must multiply every resulting component $(\lambda^{l_1+m_1+\dots})$ by

$$\frac{(l_1+m_1+\dots)!}{l_1!\,m_1!\dots}.$$

Thus $$S(\lambda^{l+m+\dots})$$

$$= \Sigma P\left\{\Sigma\left(\frac{(l_1+m_1+\dots)!}{l_1!\,m_1!\dots}\right)^{j_1}\left(\frac{(l_2+m_2+\dots)!}{l_2!\,m_2!\dots}\right)^{j_2}\dots\left(\frac{(l_\nu+m_\nu+\dots)!}{l_\nu!\,m_\nu!\dots}\right)^{j_\nu}\right\}$$
$$\times(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu}.$$

Now, $$\Sigma\frac{\{(l_1+m_1+\dots)!\}^{j_1}}{(l_1!\,m_1!\dots)^{j_1}}\frac{\{(l_2+m_2+\dots)!\}^{j_2}}{(l_2!\,m_2!\dots)^{j_2}}\dots\frac{\{(l_\nu+m_\nu+\dots)!\}^{j_\nu}}{(l_\nu!\,m_\nu!\dots)^{j_\nu}}$$
$$=\frac{(l+m+\dots)!}{l!\,m!\dots},$$

for each represents the total number of permutations of $l+m+\dots$ things, of which l are of one sort, m of a second, &c.

Hence
$$S(\lambda^{l+m+\dots}) = \Sigma\,\frac{(l+m+\dots)!}{l!\,m!\dots}\,P\,(\lambda^{l_1+m_1+\dots})^{j_1}(\lambda^{l_2+m_2+\dots})^{j_2}\dots(\lambda^{l_\nu+m_\nu+\dots})^{j_\nu}.$$

Comparing this with the known formula

$$\frac{(-)^{l+m+\dots}}{l+m+\dots}\,S(\lambda^{l+m+\dots})$$
$$=\Sigma(-)^{\Sigma j}\frac{(\Sigma j-1)!}{j_1!\,j_2!\dots j_\nu!}(\lambda^{l_1+m_1+\dots})^{j_1}(\lambda^{l_2+m_2+\dots})^{j_2}\dots(\lambda^{l_\nu+m_\nu+\dots})^{j_\nu},$$

we find, as before,

$$P=(-)^{l+m+\dots+\Sigma j}\frac{l!\,m!\dots}{(l+m+\dots-1)!}\cdot\frac{(\Sigma j-1)!}{j_1!\,j_2!\dots}.$$

It will be noticed that the general result involves only the numbers $l, m, \dots j_1, j_2, \dots$; so that, merely attending to these multiplicities, we may write the result in the hypersymbolic and compact form—

$$(-)^{l+m+\dots}\frac{(l+m+\dots-1)!}{l!\,m!\dots}S\,|lm\dots| = \Sigma(-)^{\Sigma j}\frac{(\Sigma j-1)!}{j_1!\,j_2!\dots}|j_1j_2j_3\dots|,$$

where $|j_1j_2j_3\dots|$ denotes the sum of all the corresponding separations.

This theorem enables us at once to write down an expression for the s^{th} power of the roots corresponding to every partition of s. Thus for $s = 6$, the series is

$$S(6) = (6),$$

$$S(51) = (5)(1) - (51),$$

$$S(42) = (4)(2) - (42),$$

$$S(41^2) = (4)(1)^2 - (41)(1) - (4)(1^2) + (41^2),$$

$$\tfrac{1}{2}S(3^2) = \tfrac{1}{2}(3)^2 - (3^2),$$

$$2S(321) = 2(3)(2)(1) - (32)(1) - (31)(2) - (21)(3) + (321),$$

$$S(31^3) = (3)(1)^3 - (31)(1)^2 - 2(3)(1^2)(1) + (31)(1^2) + (31^2)(1) + (3)(1^3) - (31^3),$$

$$\tfrac{1}{6}S(2^3) = \tfrac{1}{6}(2)^3 - (2^2)(2) + (2^3),$$

$$\tfrac{3}{2}S(2^21^2) = \tfrac{3}{2}(2)^2(1)^2 - 2(21)(2)(1) - (2^2)(1)^2 - (1^2)(2)^2 + (2^2)(1^2) + \tfrac{1}{2}(21)^2 + (21^2)(2) + (2^21)(1) - (2^21^2),$$

$$S(21^4) = (2)(1)^4 - (21)(1)^3 - 3(2)(1^2)(1)^2 + (21)(1)^2 + 2(1^3)(1)(2) + 2(21)(1^2)(1) + (2)(1^2)^2 - (2)(1^4) - (21)(1^3) - (21^2)(1^2) - (21^3)(1) + (21^4),$$

$$\tfrac{1}{6}S(1^6) = \tfrac{1}{6}(1)^6 - (1^2)(1)^4 + \tfrac{3}{2}(1^2)^2(1)^2 + (1^3)(1)^3 - \tfrac{1}{3}(1^2)^3 - 2(1^3)(1^2)(1) - (1^4)(1)^2 + \tfrac{1}{2}(1^3)^2 + (1^4)(1^2) + (1^5)(1) - (1^6).$$

New Tables of Symmetric Functions.

It may be gathered from the foregoing section that it is possible to form tables of symmetric functions, of a symmetrical character, corresponding to every partition of every number. We may select at pleasure any partition as the partition of restriction, and write down partitions representing every possible species of its separations; by the side of these partitions we may write down the compound symmetric functions represented by the corresponding separations. In the expansion of these compounds in a series of monomial symmetric functions, only those monomials will occur which have partitions identical with those representing the species of the separations; this follows naturally from the law of algebraic reciprocity. Thus a symmetrical table necessarily results. To make the method clear, I instance the partition

$$(21^3),$$

and exemplify, in full, the corresponding symmetric function table.

Form two columns—

(5)	(21^3)
(41)	$(21^2)(1)$
(32)	$(21)(1^2)+(1^3)(2)$
(31^2)	$(21)(1)^2$
(2^21)	$2\,(2)(1^2)(1)$
(21^3)	$(2)(1)^3$.

The left-hand column gives the species of possible separations of

$$(21^3).$$

The right-hand column gives the corresponding separations as derived from the X products (*vide* previous section).

Thus $(21)(1^2)+(1^3)(2)$ is coefficient of $x_2x_1^3$ in X_3X_2,

and $2\,(2)(1^2)(1)$ „ $x_2x_1^3$ in $X_2^2X_1$.

We may then set out in any convenient way the following table :—

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)
(21^3)						1
$(21^2)(1)$				1	2	3
$(21)(1^2)+(1^3)(2)$			1	3	2	4
$(21)(1)^2$		1	3	4	6	6
$2\,(2)(1^2)(1)$		2	2	6	4	6
$(2)(1)^3$	1	3	4	6	6	6

which reads the same by rows as by columns.

In this way we may treat every partition of every number.

We may invert these tables so as to exhibit the single partition symmetric functions in terms of compound symmetric functions symbolised by separations.

We reach then the cardinal and very important theorem of expressibility, which I now enunciate.

Theorem.—" Being given any symmetric function, of partition

$$(\lambda\mu\nu\,\ldots),$$

let $(\lambda_1\lambda_2\lambda_3 \dots)$ be any partition of λ,

$(\mu_1\mu_2\mu_3 \dots)$ " " μ,

$(\nu_1\nu_2\nu_3 \dots)$ " " ν,

... ...

Then the symmetric function

$$(\lambda\mu\nu \dots)$$

is expressible by means of compound symmetric functions which are symbolised by separations of the partition

$$(\lambda_1\lambda_2\lambda_3 \dots \mu_1\mu_2\mu_3 \dots \nu_1\nu_2\nu_3 \dots)."$$

In the example above of the partition

$$(21^3),$$

it will be noticed that there are 7 separations and 6 species of separations; there is thus

$$7-6=1,$$

syzygy between the separations.

The syzygy in question is, in fact, derivable from the separation

$$(21)(2),$$

for we may either express (21) in terms of separations of (1^3), leaving (2) unchanged, or we may leave (21) unchanged and express (2) in terms of separations of (1^2); thus the syzygy is

$$(2)\,\{(1^2)(1)-3\,(1^3)\}-(21)\,\{(1)^2-2\,(1^2)\}=0,$$

or

$$(2)(1^2)(1)-3\,(2)(1^3)-(21)(1)^2+2\,(21)(1^2)=0.$$

In general, if there are θ separations of any partition and ϕ species of separation, there must be

$$\theta-\phi$$

syzygies between the θ separations.

The h Tables direct.

	(1)
h_1	1

	(2)	(1^2)
h_2	1	1
h_1^2	1	2

	(3)	(21)	(1^3)
h_3	1	1	1
h_2h_1	1	2	3
h_1^3	1	3	6

	(4)	(31)	(2^2)	(21^2)	(1^4)
h_4	1	1	1	1	1
h_3h_1	1	2	2	3	4
h_2^2	1	2	3	4	6
$h_2h_1^2$	1	3	4	7	12
h_1^4	1	4	6	12	24

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
h_5	1	1	1	1	1	1	1
h_4h_1	1	2	2	3	3	4	5
h_3h_2	1	2	3	4	5	7	10
$h_3h_1^2$	1	3	4	7	8	13	20
$h_2^2h_1$	1	3	5	8	11	18	30
$h_2h_1^3$	1	4	7	13	18	33	60
h_1^5	1	5	10	20	30	60	120

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
h_6	1	1	1	1	1	1	1	1	1	1	1
h_5h_1	1	2	2	3	2	3	4	3	4	5	6
h_4h_2	1	2	3	4	3	5	7	6	8	11	15
$h_4h_1^2$	1	3	4	7	4	8	13	9	14	21	30
h_3^2	1	2	3	4	4	6	8	7	10	14	20
$h_3h_2h_1$	1	3	5	8	6	12	19	15	24	38	60
$h_3h_1^3$	1	4	7	13	8	19	34	24	42	72	120
h_2^3	1	3	6	9	7	15	24	21	33	54	90
$h_2^2h_1^2$	1	4	8	14	10	24	42	33	58	102	180
$h_2h_1^4$	1	5	11	21	14	38	72	54	102	192	360
h_1^6	1	6	15	30	20	60	120	90	180	360	720

The h Tables—inverse.

	h_1
(1)	1

	h_2	h_1^2
(2)	2	−1
(1^2)	−1	1

	h_3	h_2h_1	h^3
(3)	3	−3	1
(21)	−3	4	−2
(1^3)	1	−2	1

	h_4	h_3h_1	h_2^2	$h_2h_1^2$	h_1^4
(4)	4	−4	−2	4	−1
(31)	−4	7	2	−7	2
(2^2)	−2	2	3	−4	1
(21^2)	4	−7	−4	10	−3
(1^4)	−1	2	1	−3	1

	h_5	h_4h_1	h_3h_2	$h_3h_1^2$	$h_2^2h_1$	$h_2h_1^3$	h_1^5
(5)	5	−5	−5	5	5	−5	1
(41)	−5	9	5	−9	−7	9	−2
(32)	−5	5	11	−8	−11	10	−2
(31^2)	5	−9	−8	12	10	−13	3
(2^21)	5	−7	−11	10	14	−14	3
(21^3)	−5	9	10	−13	−14	17	−4
(1^5)	1	−2	−2	3	3	−4	1

	h_6	h_5h_1	h_4h_2	$h_4h_1^2$	h_3^2	$h_3h_2h_1$	h_3h_1	h_2^3	$h_2^2h_1^2$	$h_2h_1^4$	h_1^6
(6)	6	−6	−6	6	−3	12	−6	2	−9	6	−1
(51)	−6	11	6	−11	3	−17	11	−2	14	−11	2
(42)	−6	6	14	−10	3	−20	10	−6	19	−12	2
(41^2)	6	−11	−10	15	−3	21	−15	4	−20	16	−3
(3^2)	−3	3	3	−3	6	−15	6	−1	9	−6	1
(321)	12	−17	−20	21	−15	61	−30	8	−48	34	−6
(31^3)	−6	11	10	−15	6	−30	19	−4	26	−21	4
(2^3)	2	−2	−6	4	−1	8	−4	4	−10	6	−1
(2^21^2)	−9	14	19	−20	9	−48	26	−10	46	−33	6
(21^4)	6	−11	−12	16	−6	34	−21	6	−33	26	−5
(1^6)	−1	2	2	−3	1	−6	4	−1	6	−5	1