THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS OF FINITE NON-ZERO ORDER

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1. In three memoirs* which have been recently published I have investigated the asymptotic expansions of the logarithms of integral functions of finite order, and suggested that such investigations may be regarded as preliminary to a classification of integral functions. The expansions were obtained for functions of simple and multiple† linear sequence, and it was shewn that expansions for similar functions with certain types of repeated sequence could be deduced: such deductions were made in certain cases.

The investigation was based entirely on the theory of divergent series: in the first memoir I attempted to develop the theory of Borel for this purpose. Throughout the investigation no attempt was made to determine remainders for the asymptotic expansions. The fundamental procedure consisted in applying the asymptotic expansions of the Maclaurin sum-formula to a transformation by logarithmic expansion of the function investigated. The terms of the double series which arose in this way were rearranged, and were then summed by an application of Fourier's series. In order to make the application, it was assumed that $|z|$, when $|z|$ is large, was of a limited type of number, and a further assumption was made that this limitation could not affect the validity of the result; that, in fact, the form of the asymptotic expansion did not depend on the arithmetic nature of $|z|$. This assumption is valid in the case of functions of finite (non-zero) order.

It seems, however, advisable to undertake the investigation from another point of view. The theory of divergent series is but little known:

* "A Memoir on Integral Functions," *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411-500; "The Classification of Integral Functions," *Camb. Phil. Trans.*, Vol. XIX., pp. 322-355; "The Asymptotic Expansion of Integral Functions of Multiple Linear Sequence," *Camb. Phil. Trans.*, Vol. XIX., pp. 426-439.

† For the definition of these terms see § 2 of the present memoir.

parts of the theory are still obscure: it is desirable to place an important series of expansions on a basis which will appeal to mathematicians accustomed to the older methods of analysis. More than this, at the time when the former memoirs were written, I had not developed the theory of Maclaurin expansions, and it was impossible always to assign definitely the range of validity of the results. After the investigation of the previous memoir this can now be done.

The procedure employed in the present paper is that which I have previously used* to obtain the asymptotic expansions of the simple and multiple gamma functions. It is an application of Cauchy's theory of residues suggested by a noteworthy investigation of Mellin,† and afterwards applied by him to the case of the simple gamma function in a memoir‡ which has priority to my own, but of the existence of which I was ignorant when my results were being obtained. Mellin has subsequently considered§ some of the problems of the present paper: the reader may with advantage compare his investigations with my own.

2. It is convenient to repeat at the outset certain definitions which I have introduced in connection with the classification of integral functions.

A *simple* integral function is one which may be expressed as a single Weierstrassian product whose n-th zero a_n depends solely upon n and definite constants, and which is such that the law of dependence of a_n upon n is the same for all but a finite number of zeros. The function is called a *non-repeated* function if the n-th primary factor of Weierstrass's product does not correspond to a zero of order depending upon n. If there is such dependence, it is called a *repeated* simple integral function. The zero is said to be *algebraically* repeated if the number which expresses the repetition is a polynomial in n.

Functions of *multiple linear* sequence are functions whose general zero is of the type $f(a+n_1\omega_1+\ldots+n_r\omega_r)$, a and the ω's being constants and the n's being the integers which define the particular zero.

The *order* of a simple non-repeated integral function whose n-th zero is a_n is the number ρ such that $\sum\limits_n \frac{1}{|a_n|^{\rho-\epsilon}}$ is divergent, and $\sum\limits_n \frac{1}{|a_n|^{\rho+\epsilon}}$ is

* *Messenger of Mathematics*, Vol. XXIX., Part 4; *Phil. Trans. Roy. Soc.* (A), Vol. 196, Part 5; *Camb. Phil. Trans.*, Vol. XIX., §§ 55–57.

† *Acta Soc. Sci. Fennicæ*, T. XX., No. 12.

‡ *Ibid.*, T. XXIV., No. 10.

§ *Ibid.*, T. XXIX., No. 4.

convergent, however small the real positive quantity ϵ may be. If $a_n = n^\rho$, the order is $1/\rho$.

The *order* of the r-ple non-repeated function whose general zero is $f(a+\Omega)$, where $\Omega = n_1\omega_1 + \ldots + n_r\omega_r$, is similarly a number ρ such that $\sum_{n_1,\ldots,n_r} \frac{1}{|f(a+\Omega)|^{\rho-\epsilon}}$ is divergent and $\sum_{n_1,\ldots,n_r} \frac{1}{|f(a+\Omega)|^{\rho+\epsilon}}$ is convergent. When $f(x) = x^k$, the order is r/k.

3. In the present paper I consider only integral functions of finite non-zero order. We consider first simple non-repeated functions. The three standard functions of this type are

$$ {}_1P_\rho(z) = \prod_{n=0}^{\infty} \left\{ 1 + \frac{z}{(a+n\omega)^\rho} \right\}, $$

where $\rho > 1$;

$$ {}_1Q_\rho(z) = \prod_{n=0}^{\infty} \left\{ \left[1 + \frac{z}{(a+n\omega)^\rho} \right] \exp \left[\sum_{s=1}^{p} \frac{(-z)^s}{s(a+n\omega)^{s\rho}} \right] \right\}, $$

where $\rho < 1$ and $p+1 > 1/\rho > p$;

$$ {}_1R_\rho(z) = \prod_{n=0}^{\infty} \left\{ \left[1 + \frac{z}{(a+n\omega)^\rho} \right] \exp \left[\sum_{s=1}^{1/\rho} \frac{(-z)^s}{s(a+n\omega)^{s\rho}} \right] \right\}, $$

where $1/\rho$ is an integer $\geqslant 1$.

The first function is of order $1/\rho$ less than unity. The second is of non-integral order $1/\rho$ greater than unity. And the third is of integral order $1/\rho$ equal to or greater than unity. These functions are the prototypes of general simple non-repeated integral functions of finite non-zero order.

I proceed in the first place to obtain asymptotic expansions of their logarithms and to establish the conditions under which such expansions are valid.

4. THEOREM I.—*If ρ be real and positive, and if z^s has its principal value with respect to the quantity $-\omega^\rho$, the integral*

$$ \frac{1}{2\pi\iota} \int \frac{\zeta(\rho s, a)}{s} \frac{\pi z^s}{\sin \pi s} \, ds $$

vanishes when taken along any part of the great circle at infinity for which $R(s)$ is finite, provided

(1) $\quad |\arg (a/\omega)^\rho| = \pi - \epsilon,\quad$ *where* $\quad 0 < \epsilon \leqslant \pi$.

(2) $\quad |\arg (z/\omega^\rho)| = \epsilon',\quad$ *where* $\quad 0 \leqslant \epsilon' < \epsilon$.

It will also vanish along that part of the same circle for which $R(s)$ is positive and very large, provided in addition

(3) $$|z/(a+n\omega)^\rho| < 1 \quad (n = 0, 1, \ldots, \infty),$$

(4) *the circle pass between the points n, $n+1, \ldots$, n being a large positive integer.*

The integral may be written

$$\frac{1}{2\pi\iota}\int \frac{\omega^{\rho s}\,\zeta(\rho s, a)}{s}\,\frac{\pi}{\sin \pi s}\left(\frac{z}{\omega^\rho}\right)^s ds,$$

and the proof of the theorem follows the lines of Theorems I., II., and III. of the previous paper.

By saying that z^s has its principal value with respect to the quantity $-\omega^\rho$ we mean that

$$z^s = \exp\{s \log z\} = \exp\{s \log|z| + s \arg z\},$$

where $\arg z$ lies in value between $\rho \arg \omega \pm \pi$, so that

$$-\pi < \arg(z/\omega^\rho) < \pi.$$

We do not necessarily mean that z^s has its principal value with respect to the axis to the point which represents $-\omega^\rho$ in the Argand diagram.

5. Let L_0 be a contour embracing the positive half of the real axis and cutting it between $s = 1/\rho$ and $s = 1$, and let L_1 be a contour parallel to the imaginary axis and cutting the real axis in the same point as the contour L_0. Further, let L_2 be a contour parallel to L_1 cutting the real axis between $s = -l$ and $s = -(l+1)$. These contours may be compared with those drawn in § 7 of the previous paper.

THEOREM II.—*Provided the conditions* (1) *and* (2) *of* § 4 *hold,*

$$\log {}_1P_\rho(z) = -\frac{1}{2\pi\iota}\int_{L_1} \frac{\zeta(\rho s, a)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

By Cauchy's theorem, provided all four conditions of § 4 hold good,

$$I_{L_1} = I_{L_0},$$

where I denotes the integral under consideration.

The residue of I_{L_0} at $s = k$ is $(-)^k k^{-1}\zeta(\rho k, a) z^k$.

Hence $$-I_{L_0} = -\sum_{k=1}^{\infty}\frac{(-z)^k}{k}\sum_{n=0}^{\infty}\frac{1}{(a+n\omega)^{\rho k}} \quad (\text{since } \rho > 1)$$

$$= -\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-z)^k}{k(a+n\omega)^{\rho k}},$$

the inversion of the double series being legitimate under the condition (3).

Thus $$-I_{L_1} = -I_{L_0} = \log \prod_{n=0}^{\infty} \left\{1+\frac{z}{(a+n\omega)^\rho}\right\},$$

or $$-I_{L_1} = \log {}_1P_\rho(z),$$

provided the conditions (1), (2), (3), and (4) are satisfied.

But the condition (4) obviously cannot affect this equality; neither, since each expression is one-valued, continuous, and analytic for all values of $|z|$, can the condition (3).

We thus have the theorem stated.

6. THEOREM III.—*Provided the conditions* (1) *and* (2) *of* § 4 *hold good,*

$$\log {}_1P_\rho(z) = -I_{L_2} + \frac{\pi z^{1/\rho}}{\omega \sin \pi/\rho} + \zeta(0, a)\log z + \rho\zeta'(0, a)$$
$$+\sum_{n=1}^{l} \frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a),$$

where $\zeta'(0, a)$ *denotes* $\left[\frac{\partial}{\partial s}\zeta(s, a)\right]_{s=0}$ *and is equal to* [*G.F.*, p. 102]

$$\log \frac{\Gamma_1(a)}{\rho_1(\omega)} = \log\left\{\frac{\Gamma_1(a)}{\sqrt{2\pi/\omega}}\right\}$$

in the notation of the simple gamma function of parameter ω.

By Cauchy's theorem coupled with Theorem I., we see that

$$-I_{L_1} = -I_{L_2}$$

together with the sum of the residues of the subject of integration at $1/\rho$, 0, -1, ..., $-l$.

The residue of the subject of integration at $s = -n$ is

$$(-)^{n-1}\frac{\zeta(-\rho n, a)}{nz^n}$$

The residue at $s = 0$ is the absolute term in the expansion of

$$\frac{\{\zeta(0)+\rho\epsilon\zeta'(0)+\dots\}\{1+\epsilon\log z+\dots\}}{\epsilon\left\{1-\frac{\pi^2\epsilon^2}{3!}+\dots\right\}},$$

and is therefore $\zeta(0, a)\log z+\rho\zeta'(0, a)$, the logarithm having its principal value with respect to the quantity $-\omega^\rho$ (*vide* § 4).

The residue at $s = 1/\rho$ is the coefficient of $1/\epsilon$ in the expansion of

$$\rho\left\{\frac{1}{\rho\epsilon\omega}-\psi_1'(a)\ldots\right\}\frac{\pi z^{1/\rho}}{\sin \pi/\rho}=\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}. \qquad [G.F., \text{ p. } 95.]$$

Combining these results with Theorem II., we have the proposition stated.

7. We have now proved that when

$$(1) \quad |\arg (a/\omega)^\rho| = \pi-\epsilon, \quad \text{where } 0<\epsilon\leqslant\pi;$$

$$(2) \quad |\arg (z/\omega^\rho)| = \epsilon', \quad \text{where } 0\leqslant\epsilon'<\epsilon,$$

we have

$$\log {}_1P_\rho(z) = \zeta(0, a)\log z+\rho\zeta'(0, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s, a)}{s}\frac{\pi z^s}{\sin\pi s}ds;$$

and we have to investigate how far the conditions limit the asymptotic expansion to which this equality gives rise.

For this purpose we need the following :—

THEOREM IV.—*Under the conditions* (1) *and* (2) *coupled with*

$$|(a+k\omega)^\rho/z| < 1,$$

we have

$$\frac{1}{2\pi\iota}\int_{L_2}\frac{z^s}{s\,(a+k\omega)^{\rho s}}\frac{\pi}{\sin\pi s}ds = \sum_{n=l+1}\frac{(-)^n(a+k\omega)^{\rho n}}{nz^n}.$$

As in § 5, we prove that, under the conditions (1) and (2),

$$I_{L_1} = -\log\left\{1+\frac{z}{(a+k\omega)^\rho}\right\}.$$

Hence, as in § 6, under these same conditions,

$$I_{L_2} = I_{L_1}+\sum_{n=1}^{l}\frac{(-)^{n-1}(a+k\omega)^{n\rho}}{nz^n}+\log z-\rho\log(a+k\omega)$$

$$= -\log\left\{1+\frac{(a+k\omega)^\rho}{z}\right\}+\sum_{n=1}^{l}(-)^{n-1}\frac{(a+k\omega)^{n\rho}}{nz^n}.$$

Therefore, provided we have the original condition $|(a+k\omega)^\rho/z| < 1$,

$$I_{L_2} = \sum_{n=l+1}^{\infty}\frac{(-)^n(a+k\omega)^{\rho n}}{nz^n},$$

which is the theorem required.

Now we know that $\zeta(s,\, a+\omega)-\zeta(s,\, a) = -a^{-s}$,

so that
$$\zeta(s,\, a+k\omega) = \zeta(s,\, a)-\sum_{n=0}^{k-1}(a+n\omega)^{-s}.$$

Therefore, under the conditions (1) and (2),

$$\log {}_1P_\rho(z) = \zeta(0,\, a)\log z+\rho\zeta'(0,\, a)+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n,\, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}$$
$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\left\{\zeta(\rho s,\, a+k\omega)+\sum\limits_{m=0}^{k-1}\dfrac{1}{(a+m\omega)^{\rho s}}\right\}}{s}\frac{\pi z^s}{\sin\pi s}\,ds.$$

And therefore, provided in addition $|(a+m\omega)^\rho/z| < 1$, $m = 0, 1, \ldots, k-1$, we have

$$\log {}_1P_\rho(z) = \zeta(0,\, a)\log z+\rho\zeta'(0,\, a)+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n,\, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}$$
$$+\sum_{n=l+1}^{\infty}\frac{(-)^{n-1}}{n}\sum_{m=0}^{k-1}\frac{(a+m\omega)^{\rho n}}{z^n}-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s,\, a+k\omega)}{s}\frac{\pi z^s}{\sin\pi s}\,ds.$$

In this formula the condition (1) may be modified; for, in order that the integral last written may be convergent, it is merely necessary that $|\arg\{(a+k\omega)/\omega)^\rho\}| = \pi-\epsilon$, where $0<\epsilon\leqslant\pi$, and that then

$$|\arg(z/\omega^\rho)| = \epsilon',$$

where $0\leqslant\epsilon'<\epsilon$. But we may, by taking k large but finite, make $\arg(a/\omega+k)^\rho$ as small as we please, and then $|\arg(z/\omega^\rho)|$ may have any value $<\pi$.*

The series $\sum\limits_{n=l+1}^{\infty}\dfrac{(-)^{n-1}}{n}\sum\limits_{m=0}^{k-1}\dfrac{(a+m\omega)^{\rho n}}{z^n}$ will then be the sum of a large but finite number of remainders after l terms of convergent logarithmic expansions (provided $|z|$ be very large), and will therefore be of order less than the order $|z|^{-l}$.

Again $\dfrac{1}{2\pi\iota}\displaystyle\int_{L_2}\frac{\zeta(\rho s,\, a+k\omega)}{s}\frac{\pi z^s}{\sin\pi s}\,ds$ is, when $|z|$ is large, of order $1/|z|^{l+t}$ $(0<t<1)$.

If, then, z is not in the immediate vicinity of the zeros of ${}_1P_\rho(z)$, and if a is not such as to make any zero $-(a+n\omega)^\rho$ vanish, we have, when $|z|$ is large, the asymptotic expansion

$$\log {}_1P_\rho(z) = \zeta(0,\, a)\log z+\rho\zeta'(0,\, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}+\sum_{n=1}^{l}\frac{(-)^{n-1}}{nz^n}\zeta(-\rho n,\, a)+J_l,$$

where $|J_l|$ is of order less than $|z|^{-l}$.

* There is the obvious restriction that $a+n\omega$ must not vanish for positive integral values of n.

This result is a slight generalisation of that obtained by me previously by the theory of divergent series [*I.F.*, § 52]. In the present instance we have determined the remainder in the asymptotic expansion of the standard function of simple non-repeated sequence of order less than unity. The result accords with my general theorem that the asymptotic expansion is valid for all large values of $|z|$ which are not in the immediate vicinity of zeros of the function [*I.F.*, § 44].

8. It should be carefully noticed that the previous formula has only been proved under the restriction that ρ is real. In this case the zeros $-(a+n\omega)^\rho$ ultimately lie along a single line tending to infinity. But, when ρ is complex, the zeros in general cover the whole plane near infinity, for they ultimately behave like

$$(n\omega)^\rho = \exp\{\rho \log n + \rho \log \omega\},$$

and therefore each sucessive zero has a different argument. We expect, for this reason, that no asymptotic expansion will exist; and, in fact, it is easy to shew that the foregoing proof breaks down.

9. Few modifications are necessary to establish the asymptotic expansions for

$$\log {}_1Q_\rho(z) = \log \prod_{n=0}^{\infty} \left[\left\{1+\frac{z}{(a+n\omega)^\rho}\right\} \exp\left\{\sum_{s=1}^{p} \frac{(-z)^s}{s(a+n\omega)^{\rho s}}\right\}\right],$$

where $p+1 > 1/\rho > p$.

We take the contours L_0 and L_1 to cut the real axis between $s = 1/\rho$ and $s = p+1$. Then, provided the conditions (1), (2), (3), and (4) of § 4 hold good,

$$-\frac{1}{2\pi i}\int_{L_0} \frac{\zeta(\rho s, a)}{s} \frac{\pi z^s}{\sin \pi s}\, ds = -\sum_{s=p+1}^{\infty} \frac{(-z)^s}{s} \zeta(\rho s, a)$$

$$= -\sum_{s=p+1}^{\infty} \frac{(-z)^s}{s} \sum_{n=0}^{\infty} \frac{1}{(a+n\omega)^{\rho s}}$$

$$= -\sum_{n=0}^{\infty} \sum_{s=p+1}^{\infty} \frac{(-z)^s}{s(a+n\omega)^{\rho s}}$$

$$= \log {}_1Q_\rho(z).$$

Hence, by the former argument, under the conditions (1) and (2) alone,

$$-I_{L_1} = \log {}_1Q_\rho(z);$$

and therefore, by Cauchy's theorem,

$$\log {}_1Q_\rho(z) = \zeta(0)\log z+\rho\zeta'(0, a)+\frac{\pi z^{1/\rho}}{\omega\sin\pi/\rho}+\sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n}\zeta(-\rho n, a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s, a)}{s}\frac{\pi z^s}{\sin\pi s}ds,$$

where the accent in the summation denotes that the term corresponding to $n=0$ is to be omitted.

From this formula we deduce, exactly as in § 7 and under the same conditions, the asymptotic expansion of $\log {}_1Q_\rho(z)$.

10. Consider next the function

$${}_1R_\rho(z) = \prod_{n=0}^{\infty}\left[\left\{1+\frac{z}{(a+n\omega)^\rho}\right\}\exp\left\{\sum_{s=1}^{1/\rho}\frac{(-z)^s}{s(a+n\omega)^{s\rho}}\right\}\right],$$

where $1/\rho$ is an integer $\geqslant 1$.

In the case of this function we take the contours L_0 and L_1 to cut the real axis between $1/\rho$ and $1/\rho+1$.

We have, under the conditions (1) and (2) of § 4,

$$\log {}_1R_\rho(z) = -\frac{1}{2\pi\iota}\int_{L_1}\frac{\zeta(\rho s, a)}{s}\frac{\pi z^s}{\sin\pi s}ds,$$

and thence, by considering the residues at the poles of the subject of integration between L_1 and L_2, we deduce

$$\log {}_1R_\rho(z) = \rho\zeta'(0, a)+\zeta(0, a)\log z+\rho(-)^{1/\rho+1}z^{1/\rho}\psi_1^{(1)}(a)$$

$$+\rho(-)^{1/\rho}z^{1/\rho}S_1^{(2)}(a)\left\{\frac{1}{\rho}\log z-1\right\}+\sum_{n=-1/\rho+1}^{l}{}'\frac{(-)^{n-1}}{nz^n}\zeta(-n\rho, a)$$

$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{\zeta(\rho s, a)}{s}\frac{\pi z^s}{\sin\pi s}ds.$$

This is the result of making $1/\rho = p+\epsilon$ and then putting $\epsilon = 0$ in the formula for $\log {}_1Q_\rho(z)$.

For the two terms in that formula which become infinite are

$$\frac{\pi z^{p+\epsilon}}{\omega\sin\pi(p+\epsilon)}+\frac{(-)^p z^p}{p}\zeta(1-\rho\epsilon, a)$$

$$=\frac{(-z)^p}{\omega\epsilon}\{1+\epsilon\log z+\dots\}\left\{1+\frac{\pi^2\epsilon^2}{3!}+\dots\right\}+\frac{(-z)^p}{p}\left\{-\frac{1}{\omega\rho\epsilon}-\psi_1^{(1)}(a)+\dots\right\}$$

$$=\frac{(-z)^p}{\omega}\log z-\frac{(-z)^p\rho}{\omega}+\frac{(-)^{p-1}z^p}{p}\psi_1^{(1)}(a). \qquad [G.F., \text{ p. } 95.]$$

Since $S_1^{(2)}(a) = 1/\omega$ [*G.F.*, p. 80], this agrees with the former result. The calculus of limits can always be employed in this way in the more complicated formulæ to which we now proceed, when infinite terms arise for particular values of the constants involved.

11. We now proceed to consider a very general type of simple non-repeated integral function of finite (non-zero) order, and its asymptotic expansion.

Let $\phi(x)$ be a (not necessarily one-valued) function of x which is such that neither $\phi(x)$ nor its reciprocal has any singularities outside a circle of finite radius k and centre the origin, outside which $a, a+\omega, \ldots, a+n\omega, \ldots$ all lie. Further, let $\underset{|x|=\infty}{\text{Lt}}\ \phi(x) = 1$ and let ρ be a positive quantity. Then the function which, when $R(s) > 1/\rho$, is represented by*

$$\sum_{n=0}^{\infty} \frac{1}{[\phi(a+n\omega)(a+n\omega)^{\rho}]^s},$$

the many-valued functions having their principal values with respect to the axis of $-\omega$, is a function of s which has no singularities in the finite part of the plane except at the points

$$s = -(m-1)/\rho, \qquad m = 0, 1, 2, \ldots, \infty.$$

Outside the circle of finite radius k we have the expansion

$$\phi(x) = 1 + \frac{c_1}{x} + \ldots + \frac{c_m}{x^m} + \ldots,$$

and, by the conditions attached to $\phi(x)$, we have within the same region, by Abel's investigation of the binomial series,†

$$[x^{\rho}\phi(x)]^{-s} = x^{-\rho s}\left\{1 + \frac{f_1(s)}{x} + \ldots + \frac{f_m(s)}{x^m} + \ldots\right\}.$$

By the Cauchy-Hadamard theorem, when m is large, $|f_r(s)| = k^m$ to a first approximation.

We see now that, when $R(s) > 1/\rho$,

$$\sum_{m=0}^{\infty} \frac{1}{[\phi(a+n\omega)(a+n\omega)^{\rho}]^s} = \sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m, a),$$

where $f_0(s) = 1$, provided the series on the right-hand side is convergent. But, when m is large, $\zeta(\rho s+m)$ behaves like

$$\zeta(m) = \frac{(-)^m}{(m-1)!}\,\frac{d^m}{da^m}\log\Gamma_1(a);$$

* In the sequel (§ 17) this need only be true when n is large, if $a+n\omega$ never vanish and be not a singularity of $\phi(x)$ or its reciprocal.

† Abel, *Œuvres Complètes* (1881), T. I., pp. 219–238.

and, as we see by the Taylor's expansion of $\log \Gamma_1(a+t)$, the modulus of this quantity behaves like $1/\mu^m$, where μ is the minimum value of

$$|a+n\omega| \quad (n = 0, 1, 2, \ldots, \infty).$$

The series $\sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m)$ is therefore convergent with $\sum_m \left(\frac{k}{\mu}\right)^m$, that is to say, if $\mu > k$.

Thus, if $a+n\omega$ for all positive integral (including zero) values of n lies outside the circle of radius k, the function

$$Z(s, a) = \sum_{m=0}^{\infty} f_m(s)\,\zeta(\rho s+m, a)$$

is convergent and represents the continuation of the function which, when $R(s) > 1/\rho$, is represented by $\sum_{m=0}^{\infty} \frac{1}{[x\phi(x)]^s_{x=a+m\omega}}$. The sole poles of the function are at the points

$$\rho s+m = -1 \qquad \text{or} \qquad s = -(m-1)/\rho.$$

12. THEOREM.—*If* $|\arg(a/\omega)^\rho| = \pi-\epsilon$ *where* $0 < \epsilon \leqslant \pi$, *and if* $s = u+\iota v$, *then* $|Z(s)\,\omega^{\rho s}|\,e^{-(\pi-\epsilon')|v|}$, *where* $0 \leqslant \epsilon' < \epsilon$, *tends exponentially to zero as* $|v|$ *tends to infinity, u being finite.*

We have
$$Z(s) = \sum_{r=0}^{\infty} f_r(s)\,\zeta(\rho s+r).$$

Also $f_r(s)$ is an algebraic polynomial of degree r in s. Hence the $(r+1)$-th term of $Z(s)$ behaves like $(1, s)^r\,\zeta(\rho s+r)$; and therefore, when $|v|$ tends to infinity, u being finite, this term tends to infinity like

$$|\omega^{-s\rho}|\,e^{(\pi-\epsilon)|v|}.$$

Hence, as the series for $Z(s)$ remains convergent, however large $|s|$ may be, if u be finite we see that

$$|Z(s)\,\omega^{\rho s}|\,e^{-(\pi-\epsilon')|v|},$$

where $0 \leqslant \epsilon' < \epsilon$, tends exponentially to zero as $|v|$ tends to infinity.

COROLLARY.—From the formula

$$Z(s) = \sum_{n=0}^{\infty} \frac{1}{[\phi(a+n\omega)\,(a+n\omega)^\rho]^s}$$

we see that, if $u > 1/\rho$, the same expression tends, under the assigned conditions, exponentially to zero, when $|s|$ tends to infinity

13. THEOREM.—*The integral* $\frac{1}{2\pi\iota}\int \frac{Z(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds$, *in which* z^s *has its principal value with respect to the quantity* $-\omega^\rho$, *is finite when taken along*

any parallel to the imaginary axis in the finite part of the plane which does not pass through finite poles of the subject of integration, provided

(1) $$|\arg (a/\omega)^\rho| = \pi - \epsilon \quad \textit{where} \quad 0 < \epsilon \leqslant \pi,$$

(2) $$|\arg (z/\omega)^\rho| = \epsilon' \quad \textit{where} \quad 0 \leqslant \epsilon' < \epsilon.$$

The subject of integration may be written

$$\frac{\pi}{s \sin \pi s} \omega^{\rho s} Z(s) \left(\frac{z}{\omega^\rho}\right)^s,$$

and this, under the assigned conditions, tends exponentially to zero as $|v|$ tends to infinity, u being finite.

COROLLARY I.—The same integral vanishes when taken along any part of the great circle at infinity for which u is finite.

COROLLARY II.—The same integral vanishes when taken along the great circle for which u is infinite and positive, provided that, in addition to conditions (1) and (2), we have

(3) $$\left|\frac{z}{(a+n\omega)^\rho}\right| < 1 \quad (n = 0, 1, 2, \ldots, \infty),$$

(4) The circle pass between the zeros of $\sin \pi s$.

14. We now can obtain the asymptotic expansion of the logarithm of the very general simple non-repeated integral function of finite (non-zero) order,

$$F_1(z) = \prod_{n=0}^{\infty} \left[\left\{1 + \frac{z}{x^\rho \phi(x)}\right\} \exp \left\{ \sum_{s=1}^{p} \frac{(-z)^s}{s [x^\rho \phi(x)]^s} \right\} \right]_{x=a+n\omega},$$

where $p+1 > 1/\rho > p$.

For, by the method previously employed, we evidently have, under the conditions (1) and (2) of § 13,

$$\log F_1(z) = -\frac{1}{2\pi\iota} \int_{L_1} \frac{Z(s)}{s} \frac{\pi z^s}{\sin \pi s} ds,$$

where the integral is taken along a contour L_1 parallel to the imaginary axis and cutting the real axis between the points $1/\rho$ and $p+1$. Hence, if L_2 be the contour defined in § 5, cutting the real axis in $-(l+1)+\epsilon$, where this point is not a pole of the subject of integration, and ϵ is small and positive, we have, under the same conditions,

$$\log F_1(z) = -\frac{1}{2\pi\iota} \int_{L_2} \frac{Z(s)}{s} \frac{\pi z^s}{\sin \pi s} ds,$$

together with the residues of the subject of integration at its poles $s = 0$, $s = -(r-1)/\rho$, $r = 0, 1, \ldots, m$, where $(m-1)/\rho < l+1 < m/\rho$.

Now the residue at $s=0$ is the absolute term in the expansion of

$$1/\epsilon\,\{1+\epsilon\log z+\dots\}\,\{Z(0)+\epsilon Z'(0)+\dots\} = Z(0)\log z+Z'(0).$$

And the residue at $s=-(r-1)/\rho$ is*

$$\frac{\pi\rho}{1-r}\,\frac{z^{-(r-1)/\rho}}{\sin\pi\left(\frac{1-r}{\rho}\right)}\times\text{ residue of }\sum_{r=0}^{\infty}f_r(s)\,\zeta(\rho s+r)$$
$$=\frac{\pi}{\omega(1-r)}\,\frac{z^{-(r-1)/\rho}}{\sin\pi\left(\frac{1-r}{\rho}\right)}\,f_r\left(-\frac{r-1}{\rho}\right).$$

Therefore, under the conditions (1) and (2),

$$\log F_1(z) = Z(0)\log z+Z'(0)+\sum_{n=-p}^{l}{}'\,\frac{(-)^{n-1}Z(-n)}{nz^n}$$
$$+\sum_{r=0}^{m}{}''\,\frac{\pi z^{-(r-1)/\rho}}{\omega(1-r)\sin\pi\left(\frac{1-r}{\rho}\right)}\,f_r\left(\frac{1-r}{\rho}\right)-\frac{1}{2\pi\iota}\int_{L_2}\frac{Z(s)}{s}\,\frac{\pi z^s}{\sin\pi s}\,ds,$$

the double accent denoting that $r=1$ is to be excluded from the summation.

15. Suppose now that $|z|$ is very large. As in § 7, the conditions (1) and (2) can be replaced by the conditions

(1) That $a+n\omega$ does not vanish for any positive integral value of n,

(2) That $|\arg(z/\omega^\rho)|<\pi$.

The modulus of the integral along the contour L_2 may be proved to be at most of order $|z|^{-l-1+\epsilon}$.

The series $\sum_{r=0}^{\infty}\frac{\pi z^{(1-r)/\rho}f_r\left(\frac{1-r}{\rho}\right)}{(1-r)\sin\pi\left(\frac{1-r}{\rho}\right)}$ is absolutely convergent if $|z|$ be sufficiently large and differs from $\sum_{r=0}^{m}$ by a quantity which is at most of order $|z|^{-l-1}$.

For $$f_r\left(\frac{1-r}{\rho}\right)=\frac{1}{2\pi\iota}\int\frac{x^{r-1}\,dx}{[\phi(x)]^{(1-r)/\rho}}=\frac{1}{2\pi\iota}\int[x^\rho\phi(x)]^{(r-1)/\rho}\,dx,$$

the integral being taken round a circle, centre the origin and radius $k'>k$. If M be the maximum value of $[x^\rho\phi(x)]^{1/\rho}$ on this circle, the integral behaves when r is large like CM^{r-1}, where C is finite. Therefore the series is convergent if $M<|z^{1/\rho}|$.

* [*Note added April* 5, 1905.]—We assume, of course, that $(1-r)/\rho$ is not an integer. In such cases limiting forms arise.

We see, then, that, under our new conditions (1) and (2),

$$\log F_1(z) = Z(0)\log z + Z'(0) + \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z(-n) + \sum_{n=0}^{\infty}{}'' \frac{\pi z^{(1-n)/\rho} f_n\left(\frac{1-n}{\rho}\right)}{\omega(1-n)\sin\pi\left(\frac{1-n}{\rho}\right)} + J_l,$$

where J_l, when $|z|$ is large, is at most of order $|z|^{-l}$.

16. We may give a slightly more elegant form to this formula, as follows.

Suppose that $y = x^\rho \phi(x) = x^\rho \left\{1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right\}$ when $|x| > k$.

As $\arg x$ goes from 0 to 2π, $\arg y$ goes from 0 to $2\pi\rho$.

By reversion of series we obtain an expansion valid for large values of $|y|$

$$x = y^{1/\rho}\left\{1 + \frac{d_1}{y^{1/\rho}} + \frac{d_2}{y^{2/\rho}} + \dots\right\},$$

wherein also, as $\arg y$ goes from 0 to $2\pi\rho$, $\arg x$ goes from 0 to 2π.

Now we have $f_n\left(\frac{1-n}{\rho}\right) = \frac{1}{2\pi\iota}\int x^{n-1}[\phi(x)]^{(n-1)\rho}\,dx$

(the integral being taken round a circle, centre the origin and radius $> k$, from $\arg x = 0$ to $\arg x = 2\pi$)

$$= \frac{1}{2\pi\iota}\int [x^\rho \phi(x)]^{(n-1)/\rho}\,dx$$

$$= -\frac{1}{2\pi\iota}\int y^{(n-1)/\rho}\left\{\sum_{m=0}^{\infty} \frac{d_m\left(\frac{m-1}{\rho}\right)}{y^{[(m-1)/\rho]+1}}\right\}dy,$$

wherein the integral is taken along a contour which is a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$.

Now, if the integral be taken along a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$,

$$\int y^{-k/\rho-1}\,dy = r^{-k/\rho}\int_0^{2\pi\rho} \iota\,d\theta\, e^{-(k/\rho)\iota\theta} = \rho r^{-k/\rho}\int_0^{2\pi} e^{-k\iota\theta}\iota\,d\theta$$

$$= 0 \text{ if } k \neq 0, \text{ and } = 2\pi\iota\rho \text{ if } k = 0.$$

Hence

$$f_n\left(-\frac{n-1}{\rho}\right) = -\frac{1}{2\pi\iota}\,2\pi\iota\rho\, d_n\left(\frac{n-1}{\rho}\right) = -(n-1)\,d_n.$$

This relation is purely algebraical, and can be otherwise obtained by actual calculation of the quantities $f_n(s/\rho)$ and by establishing the set of relations between these quantities and the d's furnished by the reversion process.

We now see that we have the asymptotic expansion

$$\log F_1(z) = Z(0)\log z + Z'(0) + \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z(-n) + \sum_{n=0}^{\infty}{}'' \frac{\pi d_n z^{1/\rho}}{\omega \sin \pi \left(\frac{1-n}{\rho}\right) z^{n/\rho}} + J_l.$$

The quantities $Z(-n)$ which intervene may be called the Maclaurin constants.

For
$$Z(-n) = \sum_{r=0}^{\infty} f_r(-n)\,\zeta(r-\rho n),$$

and, since
$$[x^\rho \phi(x)]^n = x^{\rho n} \sum_{r=0}^{\infty} \frac{f_r(-n)}{x^n}$$

when $x = a+p\omega$ $(p = 0, 1, \ldots, m-1)$, we see that we have the Maclaurin sum-formula

$$\sum_{p=0}^{m-1} [a+p\omega)^\rho \phi(a+p\omega)]^n = Z(-n) + \sum_{p=0}^{\infty} \frac{S_p'(a)}{p!} \left[\frac{d^p}{dx^p} \int^x x^{\rho n}[\phi(x)]^n\,dx\right]_{x=m\omega}.$$

Again, we have

$$Z(0) = \sum_{r=0}^{\infty} f_r(0)\,\zeta(r) = \zeta(0) = -S_1'(a),$$

for $f_r(0) = 0$, $r \neq 0$, and $f_0(0) = 1$.

Also
$$Z'(0) = \rho\zeta'(0) + \sum_{r=0}^{\infty} f_r'(0)\,\zeta(r),$$

and therefore $-Z'(0)$ is the Maclaurin constant corresponding to the application of the Maclaurin sum-formula to the function

$$\sum_{n=0}^{m-1} \log \{x^\rho \phi(x)\}_{x=a+n\omega}.$$

17. The integral function $F_1(z)$ which has been considered is not the most general simple non-repeated function of finite non-zero order. A more general function would be*

$$\prod_{n=0}^{\infty} \left[\left\{1+\frac{z}{x^\rho \phi(x)}\right\} \exp\left\{\sum_{s=1}^{p} \frac{(-z)^s}{s[x^\rho \phi(x)]^s}\right\}\right]_{x=a+n\omega},$$

where $\phi(x)$ admits together with its reciprocal, outside a circle outside

* Verbal alterations have been made in this paragraph [*April 5th*, 1905].

which $a+n\omega$ ($n=1, 2, \ldots, \infty$) all lie, an absolutely convergent expansion of the form

$$1+\frac{c_1}{x^{\epsilon_1}}+\ldots+\frac{c_r}{x^{\epsilon_r}}+\ldots,$$

where $0<\epsilon_1<\epsilon_2\ldots$ and the ϵ's tend to infinity or are finite in number. The asymptotic expansion of the logarithm of this function can be developed in the same way. The function $F_1(z)$ which has been considered is obtained by putting $\epsilon^r=r$ ($r=1, 2, \ldots, \infty$). By a slight modification of the previous theory we may see that, provided $1/\rho$ is not an integer, provided $a+n\omega$ and $\phi(a+n\omega)$ and its reciprocal vanish for no positive integral value of n, and provided $|\arg(z/\omega^\rho)|<\pi$, we have, when $|z|$ is large, the asymptotic expansion

$$\log F_1(z)$$
$$=-S_1'(a)\log z+Z'(0)+\sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n}Z(-n)+\sum_{n=0}^{\infty}{}'' \frac{\pi z^{1/\rho}d_n}{\omega\sin\pi\left(\dfrac{1-n}{\rho}\right)z^{n/\rho}}+J_l,$$

where $p<1/\rho<p+1$, and where $|J_l|$ is at most of order $|z|^{-l}$.

When $1/\rho$ is an integer p, we may obtain the corresponding expansion by the calculus of limits, just as the expansion for ${}_1R_\rho(z)$ was deduced from ${}_1Q_\rho(z)$.

No limitation is involved in the assumption that $\underset{|x|=\infty}{\text{Lt}}\ \phi(x)=1$, for we may always ensure that this shall be the case by making the substitution $z=c_0\zeta$ in $F_1(z)$.

The result which has been obtained is, in the main, in accordance with that obtained previously for integral functions of finite non-zero order by the theory of asymptotic series [*I.F.*, §§ 53–59]. We have, however, given greater precision to the conditions under which the expansion is valid than was possible before the enumeration of the conditions under which it is legitimate to apply the Maclaurin sum-formula.

The generality of the form of the function $\phi(x)$ which we have taken is very great. We may note among special cases that $\phi(x)$ may be

(1) a rational integral function of $1/x$ which does not vanish when $1/x=0$, $R(1/x)$ say;

(2) a similar rational integral function of negative fractional powers of x;

(3) of the form $R(1/x)e^{G(1/x)}$, where $G(x)$ is an integral function of x;

(4) of the the form $R(1/x)\mu(1/x)$, where $\mu(x)$ is a meromorphic function with no poles or zeros within a circle of finite radius surrounding the origin.

We may state the final result as follows :—

The logarithm of the specified general type of integral function of finite non-zero order with a single sequence of non-repeated zeros admits, when $|z|$ is large, an asymptotic expansion valid everywhere but in the neighbourhood of the zeros of the function ; and all the coefficients of this expansion can be built up from the simple Riemann ζ function $\zeta(s, a \mid \omega)$.

If the function is of order $1/\rho$ where $p+1 > 1/\rho > p$, the dominant term of the asymptotic expansion is of the order of magnitude of $z^{1/\rho}$, and, if $1/\rho$ is an integer p, is of the order of magnitude of $z^p \log z$.

18. I will now briefly indicate the extension of the previous theory to non-repeated functions of multiple linear sequence. We base the investigation on the properties of the multiple Riemann ζ function $\zeta_r(s, a \mid \omega_1, \ldots, \omega_r)$, defined in § 15 of the previous paper, the theory of which has been developed in the author's memoir on the multiple gamma function.

Let $\phi(x)$ be the function of x defined in § 11 which is such that neither $\phi(x)$ nor its reciprocal has any singularities outside a circle of finite radius k and centre the origin, outside which $(a+\Omega)$, where $\Omega = n_1\omega_1+\ldots+n_r\omega_r$, lies when $n_1, \ldots, n_r$ have any positive integral values (zero included). We assume that the ω's all lie on the same side of some straight line P through the origin. Further, let $\underset{|x|=\infty}{\text{Lt}} \phi(x) = 1$.

Then the function which, when $R(s) > r/\rho$, is represented by

$$\sum_{n=0}^{\infty} \frac{1}{[\phi(a+\Omega)(a+\Omega)^{\rho}]^s},$$

where the many-valued functions have their principal values with respect to the axis of $-1/L$ defined in § 15 of the previous paper, is represented for all values of s by

$$Z_r(s) = \sum_{m=0}^{\infty} f_m(s)\, \zeta_r(\rho s+m, a),$$

where $\zeta_r(s, a)$ is the r-ple Riemann ζ function, and where

$$[\phi(x)]^{-s} = \sum_{m=0}^{\infty} \frac{f_m(s)}{x^m}.$$

The sole singularities of the function, *qua* function of s, in the finite part of the plane are at the points

$$\rho s+t = q, \qquad \text{where } \begin{cases} q = 1, 2, \ldots, r, \\ t = 0, 1, \ldots, \infty, \end{cases}$$

or

$$s = (q-t)/\rho,$$

For [*M.G.F.*, § 31] the sole singularities of $\zeta_r(s, a)$ are at the points $s = q$.

19. We now assume, for otherwise the points $(a+\Omega)^\rho$ will cover the whole region of the plane near infinity, that all the points Ω^ρ lie within an angle $\theta\,(< 2\pi)$ whose vertex is the origin. This necessitates that, when $\rho \geqslant 2$, the points ω lie within an angle $\theta/\rho\,(< 2\pi/\rho)$. In this latter case we take the fundamental line P (see § 15 of the previous paper) to be the external bisector of this angle.

We shall obtain the system of points Ω^ρ within the angle θ by rotating each point Ω until its argument is ρ times its former value. Let $1/L_\rho$ be the line which is obtained from $1/L$ in this way. Then, if ϕ be the argument of the line $1/L$, that of the *quantity* $1/L_\rho$ is $\rho\phi$.

Let z^s have its principal value with respect to the *quantity* $-1/L_\rho$. (1)

By this we mean that $\rho\phi-\pi < \arg z < \rho\phi+\pi$.

Further, assume that z does not lie within the region of the points $-\Omega^\rho$; (2)

so that therefore

$$|\arg z-\rho\phi| = \eta, \quad \text{where} \quad 0 \leqslant \eta < \begin{cases} \pi-\dfrac{\rho\pi}{2} \ (\rho < 2), \\ \pi-\tfrac{1}{2}\theta \ \ (\rho \geqslant 2). \end{cases}$$

In the expression for $\zeta_r(s, a)$ terms of the type $(a+\Omega)^{\rho s}$ occur, which have their principal values with respect to $-1/L$. By taking a to be positive with respect to the ω's (3), we ensure that $\arg(a+\Omega)$ differs from ϕ by a quantity $< \begin{cases}\tfrac{1}{2}\pi & (\rho < 2), \\ \tfrac{1}{2}\theta/\rho & (\rho \geqslant 2);\end{cases}$ and therefore that

$$|\arg(a+\Omega)^\rho-\rho\phi| = \epsilon, \quad \text{where} \quad 0 \leqslant \epsilon < \begin{cases}\tfrac{1}{2}\rho\pi & (\rho < 2), \\ \tfrac{1}{2}\theta & (\rho \geqslant 2).\end{cases}$$

We may therefore put under the conditions (1), (2), and (3)

$$\frac{z^s}{(a+\Omega)^{\rho s}} = (r_n e^{\iota\psi_n})^s,$$

and we shall have $\qquad 0 \leqslant |\psi_n| \leqslant \epsilon+\eta < \pi.$

If, now, we have the further condition $r_n < 1$ (4), we have the proposition *that*

$$Z_r(s)\, z^s e^{-\pi|v|},$$

where $R(s) > -(l+1)$ *and* $s = u+\iota v$, *tends exponentially to zero as* $|s|$ *tends to infinity*. This may be proved by the same methods as those already employed

Suppose now that $p+1 > r/\rho > p$. Let ${}_rL_0$ and ${}_rL_1$ be contours, similar to those employed in § 5, cutting the real axis between r/ρ and $(p+1)$. Then, by the proposition just stated, the integral

$$\frac{1}{2\pi\iota}\int \frac{Z_r(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

is finite when taken along the contours ${}_rL_0$, ${}_rL_1$, and L_2, and the two former integrals are equal to one another.

20. Let $F_r(z)$ denote the integral function of multiple linear sequence and order r/ρ

$$\prod_{n_1=0}^{\infty}\cdots\prod_{n_r=0}^{\infty}\left[\left\{1+\frac{z}{x^\rho\phi(x)}\right\}\exp\left\{\sum_{s=1}^{p}\frac{(-z)^s}{s\{x^\rho\phi(x)\}^s}\right\}\right]_{x=a+\Omega},$$

p being such that $p+1 > r/\rho > p$.

Then, as previously, we see that, provided $r_n < 1$ and $|\psi_n| < \pi$,

$$\log F_r(z) = -\frac{1}{2\pi\iota}\int_{{}_rL_1}\frac{Z_r(s)}{s}\,\frac{\pi z^s}{\sin \pi s}\,ds.$$

But both sides of the equality are one-valued continuous analytic functions for all values of r_n. Therefore we may dispense with the condition (4) of § 19.

Now apply Cauchy's theorem and change the contour of integration from ${}_rL_1$ to L_2. We get

$$F_r(z) = -\frac{1}{2\pi\iota}\int_{L_2}\frac{Z_r(s)\,\pi z^s}{s\sin\pi s}\,ds+\sum_{n=-p}^{l}{}'\,\frac{(-)^{n-1}}{nz^n}\,Z_r(-n)$$

plus the sum of the residues of the subject of integration at the points

$s = 0,$

$s = (q-t)/\rho \qquad (q = 1, 2, \ldots, r;\ \ t = 0, 1, \ldots, m),$

where $(m+1-q)/\rho > l+1 > (m-q)/\rho$.

Now the residue of $\dfrac{\pi z^s}{s\sin\pi s}\,\zeta_r(\rho s+t)$ at $s = (q-t)/\rho$ is

$$\frac{\pi z^{(q-t)/\rho}}{(q-t)\sin\pi\left(\dfrac{q-t}{\rho}\right)}\;\frac{(-)^{q+r}\,{}_rS_1^{(q+1)}(a)}{(q-1)!} \qquad [M.G.F., § 31].$$

Hence the sum of the residues of the subject of integration at the points $s = (q-t)/\rho$ is the sum of such terms of the infinite series

$$\sum_{q=1}^{r}\frac{(-)^{q+r}\pi\,{}_rS_1^{(q+1)}(a)}{(q-1)!}\sum_{t=0}^{\infty}{}''\,f_t\left(\frac{q-t}{\rho}\right)\frac{z^{(q-t)/\rho}}{(q-t)\sin\pi\left(\dfrac{q-t}{\rho}\right)}$$

as are not of order less than $1/|z|^{l+1}$ when $|z|$ is large. The double accent in the summation denotes that the term for which $t = q$ is to be omitted.

The series just written is absolutely convergent for sufficiently large values of $|z|$, since

$$f_t\left(\frac{q-t}{\rho}\right) = \frac{1}{2\pi\iota}\int \frac{x^{t-1}}{[\phi(x)]^{(q-t)/\rho}}\,dx = \frac{1}{2\pi\iota}\int x^{q-1}[x^\rho \phi(x)]^{(t-q)/\rho}\,dx,$$

the integral being taken round a circle of radius $k' > k$. Hence, when t is large,

$$\left|f_t\left(\frac{q-t}{\rho}\right)\right| < k'^q M^{(t-q)/\rho},$$

where M is the maximum value of $x^\rho \phi(x)$ on the circle in question. The series is therefore convergent if $|z| > M$.

Again, the residue of the subject of integration at $s = 0$ is the absolute term in the expansion

$$\sum_{t=0}^{\infty} \frac{\{1+\epsilon \log z+\dots\}}{\epsilon\left\{1-\frac{\pi^2\epsilon^2}{3!}+\dots\right\}}\{\zeta_r(t)+\rho\epsilon\zeta_r'(t)+\dots\}\{f_t(0)+\epsilon f_t'(0)+\dots\},$$

and is therefore

$$\sum_{t=0}^{\infty}\{f_t(0)[\zeta_r(t)\log z+\rho\zeta_r'(t)]+f_t'(0)\,\zeta_r(t)\} = \zeta_r(0)\log z+\rho\zeta_r'(0)+\sum_{t=0}^{\infty} f_t'(0)\,\zeta_r(t)$$
$$= \zeta_r(0)\log z + Z_r'(0).$$

We note [*M.G.F.*, §§ 22 and 23] that $\zeta_r(0) = (-)^r_r S_1'(a)$ and that

$$\zeta_r'(0) = \left[\frac{\partial}{\partial s}\zeta_r(s, a)\right]_{s=0} = \log\frac{\Gamma_r(a)}{\rho_r(\omega)},$$

where $\Gamma_r(a)$ is the r-ple gamma function and $\rho_r(\omega)$ is the r-ple Stirling modular form.

We have then

$$\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n) + Z_r'(0) + \zeta_r(0, a)\log z$$
$$+\sum_{q=1}^{r}\frac{(-)^{q+r}\pi_r S_1^{(q+1)}(a)}{(q-1)!}\sum_{t=0}^{\infty}{}'' f_t\left(\frac{q-t}{\rho}\right)\frac{z^{(q-t)/\rho}}{(q-t)\sin\pi\left(\frac{q-t}{\rho}\right)}$$
$$-\frac{1}{2\pi\iota}\int_{L_2}\frac{Z_r(s)\,\pi z^s}{s\sin\pi s}\,ds + J_l,$$

where J_l, when $|Z|$ is large, is of order less than $|z|^{-l}$.

21. In this equality the many-valued functions z^s, $z^{(q-t)/\rho}$, and $\log z$ have their principal values with respect to the *quantity* $-1/L_\rho$; that is to

say, in each case the argument of z lies between $\rho\phi \pm \pi$. Further, we have assumed that z does not lie within the region within which the points $-\Omega^\rho$ lie. And, finally, we have assumed that a is positive with respect to the ω's, that is to say, that a lies within the region within which the ω's lie.

Exactly as in § 7, we may remove the last restriction. And we obtain the theorem :—

If $|z|$ is large and not within that part of the region at infinity within which the points $-(a+\Omega)^\rho$ lie, and if a is of finite modulus, and not such as to make $a+\Omega$ vanish identically for any positive integral values of $n_1, \ldots, n_r$ (zero included), then

$$\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n) + Z_r'(0) + \zeta_r(0, a) \log z$$

$$+ \sum_{q=1}^{r} \frac{(-)^{q+r} \pi_r S_1^{(q+1)}(a)}{(q-1)!} \sum_{t=0}^{\infty}{}'' f_t\left(\frac{q-t}{\rho}\right) \frac{z^{(q-t)/\rho}}{(q-t) \sin \pi \left(\frac{q-t}{\rho}\right)} + J_l,$$

where $|J_l|$ is of order less than $|z|^{-l}$. In this expression

$$z^{(q-t)/\rho} = \exp\left[\frac{q-t}{\rho} \log z\right],$$

and $\log z$ is such that its argument lies between $\rho\phi \pm \pi$.

22. We may give a more elegant form to the series just written.

Let $$y = x^\rho \phi(x) = x^\rho \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots\right),$$

where, as $\arg x$ goes from 0 to 2π, $\arg y$ goes from 0 to $2\pi\rho$, and suppose that, by reversion of series, we obtain

$$x = y^{1/\rho}\left\{1 + \frac{{}_1d_1}{y^{1/\rho}} + \ldots + \frac{{}_1d_t}{y^{t/\rho}} + \ldots\right\}$$

and $$x^q = y^{q/\rho}\left\{1 + \frac{{}_qd_1}{y^{1/\rho}} + \ldots + \frac{{}_qd_t}{y^{t/\rho}} + \ldots\right\},$$

where $q = 1, 2, \ldots, r$.

Then, as formerly (§ 16), we have

$$f_t\left(\frac{q-t}{\rho}\right) = \frac{1}{2\pi\iota} \int y^{(t-q)/\rho} \frac{1}{q}\, d\left\{y^{q/\rho} + \ldots + \frac{{}_qd_t}{y^{(t-q)/\rho}} + \ldots\right\}$$

taken along a circular arc from $\arg y = 0$ to $\arg y = 2\pi\rho$.

Thus $$f_t\left(\frac{q-t}{\rho}\right) = \frac{\rho}{2\pi\iota q} \int \left(\frac{q-t}{\rho}\right) {}_qd_t \frac{dy}{y}$$

taken round a circle enclosing the origin,

$$= \frac{q-t}{q}\, {}_q d_t.$$

Hence the previous formula may be written

$$\log F_r(z) = \sum_{n=-p}^{l}{}' \frac{(-)^{n-1}}{nz^n} Z_r(-n) + (-)^r {}_rS_1'(a) \log z + Z_r'(0)$$
$$+ \sum_{q=1}^{r} \frac{(-)^{q+r} \pi\, {}_rS_1^{(q+1)}(a)\, z^{q/\rho}}{q!} \sum_{t=0}^{\infty}{}'' \frac{{}_q d_t}{z^{t/\rho} \sin \pi \left(\frac{q-t}{\rho}\right)} + J_l.$$

If we make l infinite, we obtain the asymptotic expansion of $\log F_r(z)$, which is valid when $|z|$ is very large and not within the domain of the large zeros of $F_r(z)$. If these zeros cover the whole plane at infinity, no asymptotic expansion exists. The formula now obtained is a development of the formula previously adumbrated [*Integral Functions of Multiple Linear Sequence*, §§ 11 and 15]. The asymptotic expansions of the general integral functions* of multiple linear sequence, non-repeated zeros, and finite non-zero order follow in the same way.

In obtaining the previous formula we have assumed that $1/\rho$ is not an integer, and that $(q-t)/\rho$ is not an integer for the values

$$q = 1, 2, \ldots, r; \qquad t = 0, 1, \ldots, \infty \quad (q \neq t).$$

When such exceptional cases arise we can always obtain a definite formula by the use of the calculus of limits.

23. As particular cases of the general result just obtained we may write down the asymptotic expansions of the three standard functions of finite non-zero order and multiple linear sequence:—

$${}_rP_\rho(z) = \prod_{n_1=0}^{\infty} \cdots \prod_{n_r=0}^{\infty} \left\{1 + \frac{z}{(a+\Omega)^\rho}\right\} \quad (\rho > r),$$

$${}_rQ_\rho(z) = \prod_{n_1=0}^{\infty} \cdots \prod_{n_r=0}^{\infty} \left[\left(1+\frac{z}{x^\rho}\right) \exp \sum_{s=1}^{p} \frac{(-z)^s}{sx^{p\rho}}\right]_{x=a+\Omega},$$

where p is an integer such that $p+1 > r/\rho > p$,

$${}_rR_\rho(z) = \prod_{n_1=0}^{\infty} \cdots \prod_{n_r=0}^{\infty} \left[\left(1+\frac{z}{x^\rho}\right) \exp \sum_{s=1}^{p} \frac{(-z)^s}{sx^{p\rho}}\right]_{x=a+\Omega},$$

where r/ρ is an integer p.

* Such as correspond to those mentioned in § 17 [*April* 5, 1905].

We have

$$\log {}_rP_\rho(z) = \sum_{n=1}^{l} \frac{(-)^{n-1}}{nz^n} \zeta_r(-\rho n, a) + \rho \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z$$
$$+ \sum_{q=1}^{r} \frac{(-)^{q+r} \pi {}_rS_1^{(q+1)}(a)}{q!} \frac{z^{q/\rho}}{\sin \frac{\pi q}{\rho}} + J_l,$$

agreeing with *I.F.M.L.S.*, § 5.

We have a similar expansion for $\log {}_rQ_\rho(z)$, except that the first series is summed from $n = -p$ to l (zero excluded).

To obtain the asymptotic expansion for $\log {}_rR_\rho(z)$ we apply the calculus of limits to the formula for $\log {}_rQ_\rho(z)$. The work has been carried out [*I.F.M.L.S.*, §§ 16 and 17]. The result is that, if p be not a multiple of r,

$$\log {}_rR_\rho(z) = \frac{r}{\rho} \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z + \sum_{s=-p+1}^{l}{}' \frac{(-)^{s-1}}{sz^s} \zeta_r\left(-\frac{sr}{p}, a\right)$$
$$+ \frac{(-)^{p+r} z^p}{p(r-1)!} \psi_r^{(r)}(a) + \frac{(-z)^p}{(r-1)!} {}_rS_1^{(r+1)}(a) \left\{\frac{\log z}{r} - \sum_{k=1}^{r} \frac{1}{kp}\right\}$$
$$+ \sum_{m=1}^{r-1} \frac{(-)^{m+r} \pi z^{mp/r} {}_rS_1^{(m+1)}(a)}{m! \sin \frac{mp\pi}{r}} + J_l.$$

If $p = kr$, where k is an integer, we have

$$\log {}_rR_\rho(z) = \frac{r}{p} \log \frac{\Gamma_r(a)}{\rho_r(\omega)} + (-)^r {}_rS_1'(a) \log z + \sum_{s=-p}^{l}{}^{*} \frac{(-)^{s-1}}{sz^s} \zeta_r\left(-\frac{sr}{p}, a\right)$$
$$+ \sum_{m=1}^{r} \frac{(-)^{(k+1)m} z^{km}}{m!\, k} \psi_r^{(m)}(a) + \sum_{m=1}^{r} \frac{(-)^{(k+1)m+r}}{m!\, k} {}_rS_1^{(m+1)}(a) \left\{k \log z - \sum_{n=1}^{m} \frac{1}{n}\right\} + J_l,$$

the star indicating that the terms which correspond to $s = 0, -k, -2k, \ldots, -rk$ are to be omitted in the summation.