200. A New Method of Finding the Condition That the General Conicoid Should Be One of Revolution and of Finding the Equation of Its Axis

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Source: *The Mathematical Gazette*, Vol. 3, No. 57 (May, 1906), pp. 326-327
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3605001
Accessed: 09-01-2016 10:11 UTC

Putting each of these three equal quantities $=k$, and eliminating λ_1 and k from the three linear equations thus formed, we have for the equation of the axes

$$\begin{vmatrix} 2bc(w'\alpha+v\beta+u'\gamma)(v'\alpha+u'\beta+w\gamma)-c^2(w'\alpha+v\beta+u'\gamma)^2 - b^2(v'\alpha+u'\beta+w\gamma)^2, & 2bcu'-c^2v-b^2w, & 1 \\ 2ca(v'\alpha+u'\beta+w\gamma)(u\alpha+w'\beta+v'\gamma)-a^2(v'\alpha+u'\beta+w\gamma)^2 - c^2(u\alpha+w'\beta+v'\gamma)^2, & 2cav'-a^2w-c^2u, & 1 \\ 2ab(u\alpha+w'\beta+v'\gamma)(w'\alpha+v\beta+u'\gamma)-b^2(u\alpha+w'\beta+v'\gamma) - a^2(w'\alpha+v\beta+u'\gamma)^2, & 2abw'-b^2u-a^2v, & 1 \end{vmatrix}=0.$$

H. L. TRACHTENBERG.

200. [L². 2. e; 4. a.] *A new method of finding the condition that the general conicoid should be one of revolution and of finding the equation of its axis.*

$$\{(ax_0+hy_0+gz_0+u)x+(hx_0+by_0+fz_0+v)y + (gx_0+fy_0+cz_0+w)z+(ux_0+vy_0+wz_0+d)\}^2 = \lambda[ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d]$$

is the system of conicoids touching

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2fy+2wz+d=0$$

round the curve of section by the polar plane of $x_0y_0z_0$. If (1) the surface is one of revolution *and* (2) $x_0y_0z_0$ is on the axis of revolution, and under these conditions alone, one of the conicoids of this system will be a sphere, for all the tangents from $x_0y_0z_0$ to the conicoid are equal. Let λ_1 be the value of λ for this sphere. Then writing down the conditions for a sphere

$$\left.\begin{aligned} &(ax_0+hy_0+gz_0+u)^2-\lambda a=(hx_0+by_0+fz_0+v)^2-\lambda b=(gx_0+fy_0+cz_0+w)^2-\lambda c,\\ &(hx_0+by_0+fz_0+v)(gx_0+fy_0+cz_0+w)-\lambda f=0,\\ &(gx_0+fy_0+cz_0+w)(ax_0+hy_0+gz_0+u)-\lambda g=0,\\ &(ax_0+hy_0+gz_0+u)(hx_0+by_0+fz_0+v)-\lambda h=0. \end{aligned}\right\}$$

Thus, eliminating λ, the axis lies on all the following surfaces.

$$(\alpha)\ldots\ldots\ldots\ldots\left\{\begin{aligned} &\frac{(hx+by+fz+v)^2-(gx+fy+cz+w)^2}{b-c}\\ &=\frac{(gx+fy+cz+w)^2-(ax+hy+gz+u)^2}{c-a}\\ &=\frac{(ax+hy+gz+u)^2-(hx+by+fz+v)^2}{a-b}\\ &=\frac{(hx+by+fz+v)(gx+fy+cz+w)}{f}\\ &=\frac{(gx+fy+cz+w)(ax+hy+gz+u)}{g}\\ &=\frac{(ax+hy+gz+u)(hx+by+fz+v)}{h}. \end{aligned}\right.$$

Now three of these surfaces are the three pairs of planes

$$(ax+hy+gz+u)\{h(gx+fy+cz+w)-g(hx+by+fz+v)\}=0.$$
$$(hx+by+fz+v)\{f(ax+hy+gz+u)-h(gx+fy+cz+w)\}=0.$$
$$(gx+fy+cz+w)\{g(hx+by+fz+v)-f(ax+hy+gz+u)\}=0.$$

Call these $\phi_1 . F_1=0$, $\phi_2 . F_2=0$, $\phi_1 . F_3=0$. If we can find one line and one only, that is common to all three pairs, that line must be the axis. Now the planes $F_1=0$, $F_2=0$, $F_3=0$ have a common line for $F_1+F_2+F_3\equiv 0$ and *in general* no other three planes, one from each pair, have a common line. Thus in general the axis is given by

$$(\beta) \ldots\ldots\ldots f(ax+hy+gz+u)=g(hx+by+fz+v)=h(gx+fy+cz+w).$$

Putting each of these $=K$ and solving the equations we have

$$\frac{x}{\begin{vmatrix} h & g & u-K/f \\ b & f & v-K/g \\ f & c & w-K/h \end{vmatrix}}=\frac{y}{\begin{vmatrix} g & a & u-K/f \\ f & h & v-K/g \\ c & g & w-K/h \end{vmatrix}}=\frac{z}{\begin{vmatrix} a & h & u-K/f \\ h & b & v-K/g \\ g & f & w-K/h \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}}.$$

Thus the equation of the axis is

$$\frac{x-\frac{U}{D}}{A'/f+H'/g+G'/h}=\frac{y-\frac{V}{D}}{H'/f+B'/g+F'/h}=\frac{z-\frac{W}{D}}{G'/f+F'/g+C'/h}.^*$$

Also using the fact (β) that

$$ax+hy+gz+u : hx+by+fz+v : gx+fy+cz+w=1/f : 1/g : 1/h$$

we get from (α)

$$\frac{1/g^2-1/h^2}{b-c}=\frac{1/h^2-1/f^2}{c-a}=\frac{1/f^2-1/g^2}{a-b}=\frac{1}{fgh},$$

$$\text{i.e. } b-c=\frac{hf}{g}-\frac{fg}{h} \text{ and two similars,}$$

$$\text{i.e. } \frac{gh}{f}-a=\frac{hf}{g}-b=\frac{fg}{h}-c$$

are the conditions for a surface of revolution. H. L. TRACHTENBERG.

Prof. E. B. Elliott's note (*Gazette*, Oct. 1905, p. 236) seems fallacious. To show this, suppose that the convergent sequence $s_1, s_2, \ldots$, of rationals tend to *no rational limit*. Then, given any positive rational ϵ', there is a (one or more) rational s such that, for some n and every p, $|s-s_{n+p}|<\epsilon'$. But, as ϵ' is diminished to zero, we can find *no* rational s valid for *all* such ϵ's. Thus the conclusion of (1) fails. The fact is, that logically a definition of a real number (*i.e.*, a sequence $s_1, s_2, \ldots$, of rationals such that, where ϵ is any positive rational, there is an integer n, which depends on ϵ, such that, for every p, $|s_n-s_{n+p}|<\epsilon$) precedes the criterion for the existence of a limit, and that then the sufficiency of this criterion follows at once (cf. *Encykl. der Math. Wiss.*, I. A 3, §§ 5, 6, 12, 13); though geometrical ways of thinking obscured this logical fact, even with Cauchy (cf. my remarks in *Bibl. Math.*, 1905, p. 206), until the time of Weierstrass.

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* The dashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

The undashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$.