Construction of Conic Sections by Paper-folding*

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A METHOD has been described[1] for constructing a parabola as the envelope of the creases formed on folding a piece of paper in such manner that a fixed point always falls upon a fixed straight line.

The other conic sections also can be similarly obtained, if for the straight line a circle is substituted[2], as is shown by the accompanying examples (Figs. 2 and 3) and by the following analytical demonstration.

Referring to Fig. 3, let C be the center of the fixed circle, and P the fixed point.

Bisect $C\,P$ in ●, and make ● the origin of a system of rectangular co-ordinates, with ● P for X axis. Let $O\,P = x_o$.

Then the paper is so folded that P falls upon some point Q of the circle.

If $x_1 y_1$ are the co-ordinates of Q we have:

$$(x_1 + x_o)^2 + y_1^2 = R^2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where R is the radius of the fixed circle.

If $x_2 y_2$ are the co-ordinates of the mid-point S of $P\,Q$, then

$x_1 = 2x_2 - x_o$

$y_1 = 2y_2$

Substituting these values in (1.) and simplifying:

$$x_2^2 + y_2^2 = \frac{R^2}{4} = r^2 \text{ say}$$

i. e., the point S lies upon a circle having its center at O, and a radius $r = \frac{R}{2}$.

Now the crease produced is evidently $R\,T$, perpendicular to $P\,O$ in S. Its equation is

$$y - y_2 = \frac{(x_o - x_2)}{y_2} \cdot (x - x_2).$$

Rearranging and putting r^2 for $(x_2^2 + y_2^2)$,

$$x_2(x + x_o) = r^2 + x x_o - y y_2.$$

Squaring

$$x_2^2 (x + x_o)^2 = r^4 + 2r^2 x x_o + x^2 x_o^2 - 2y y_2 (r^2 + x x_o) + y^2 y_2^2.$$

Putting $x_2^2 = r^2 - y_2^2$, simplifying, and arranging as a quadratic in y_2

$$y_2^2 \left\{ y^2 + (x + x_o)^2 \right\} - 2y_2 y (r^2 + x x_o) + r^2 (r^2 - x^2 - x_o^2) + x^2 x_o^2 = 0 \ldots\ldots\ldots\ldots (2)$$

For a given pair of values of x and y, equation (2), gives either two complex, two real and different, or two real and equal roots for y_2.

This means that the number of creases which can be produced through a given point in the manner specified is either 0, 2 or 1.

The points through which only one crease can be made evidently lie on the envelope of the series of creases, the single crease through each being tangent at such point to the envelope, so that the condition for this envelope is given by the condition for the equality of the roots of (2), viz:

$$4y^2 (r^2 + x x_o)^2 - 4 \left\{ y^2 + (x + x_o)^2 \right\} \left\{ r^2 (r^2 - x^2 - x_o^2) + x^2 x_o^2 \right\} = 0$$

$$r^2 y^2 + x^2 (r^2 - x_o^2) - r^2 (r^2 - x_o^2) = 0$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2 - x_o^2} = 1.$$

Hence the envelope is an ellipse or a hyperbola, according as $r > x_o$ or $r < x_o$, i. e., according as P lies within the fixed circle, or outside the same.

The special cases $x_o = 0$ and $r = \infty$ may be left for discussion by the reader, who may also, as an exercise, work out a purely geometrical proof of the construction here given.

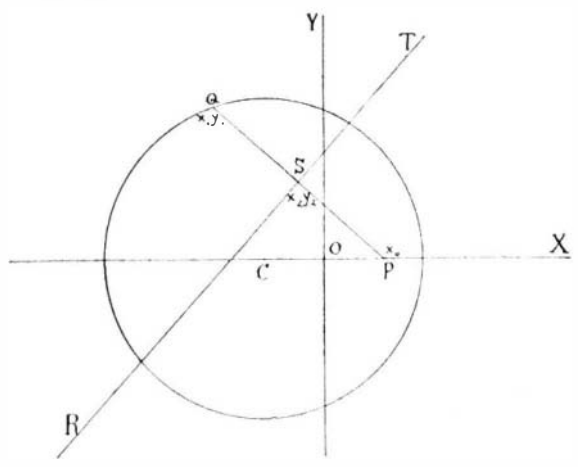

Fig. 1.

* Reproduced from *School Science and Mathematics*.

[1] S. Row (W. W. Beman and D. E. Smith) Geometric Exercises in Paper-Folding, 1901, p. 116. See also Hardcastle, Jn. Brit. Astron. Assoc., May, 1910.

[2] It is, of course, necessary to use translucent paper (tracing paper), or, if using opaque paper, to mark the fixed point on the back of the sheet, and on the edge of a perforation made in the same.

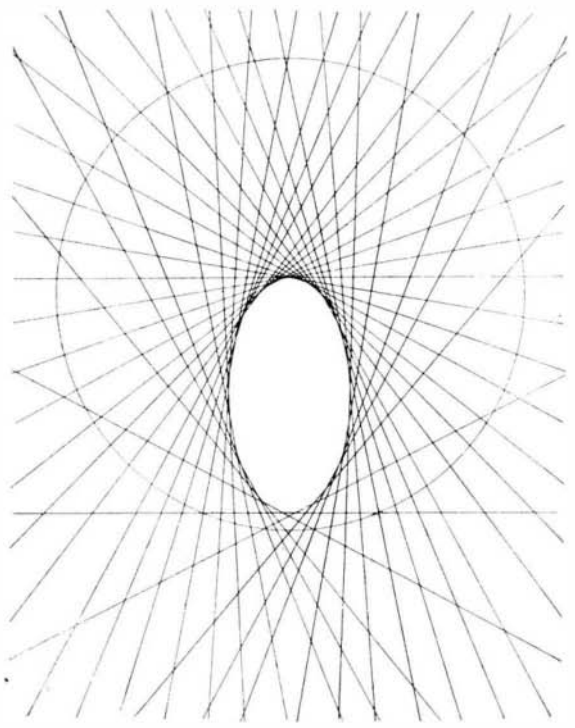

Fig. 2.

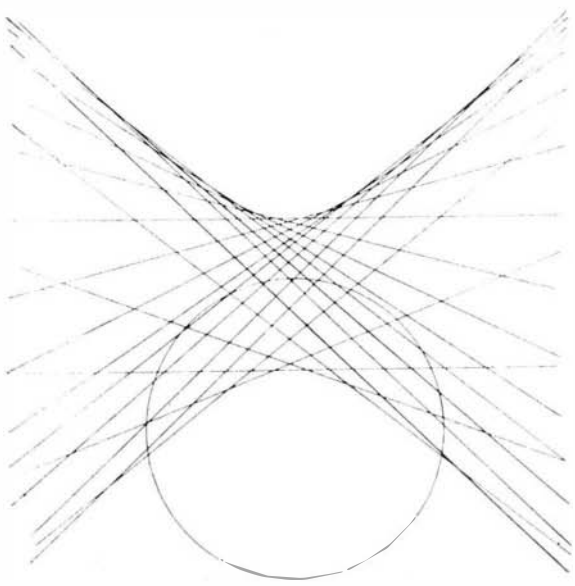

Fig. 3.

Science Notes

The First Eraser.—According to *Gummi-Zeitung*, India rubber was used for the first time as an eraser in 1770, when a very small piece was sold for three marks. It was, however, many years later before it was put into general use. Prior to this the crumb of bread was used for erasing purposes.

Preservation of Timber in the United States.—In 1910 over sixty-three million gallons of creosote and nearly seventeen million pounds of chloride of zinc were used in preserving timber in the United States. There were also used small quantities of corrosive sublimate, water gas tar, crude oil and refined coal tar.—*Wood Craft.*

Metallic Deposit on Aluminium.—The bath used for this purpose consists of a solution of anhydrous salts of copper, nickel or tin in anhydrous wood alcohol. The aluminium to be coated has to be rubbed off with some fatty or greasy metallic polish, the fat to serve as a protection against oxidation. It is then dipped into the solution mentioned. The alcohol dissolves the greasy substance, and the copper, nickel or tin forms a coating which adheres firmly to the aluminium.—*Chem. Tech. Rep.*

The Chemical Composition of Sauerkraut.—Four different samples of sauerkraut have been subjected to a chemical analysis with the following results: Water, 88.00-90.84 per cent; protein, 1.39-1.68 per cent; fat, 0.28-0.38 per cent; lactic acid, 1.22-1.78 percent; dextrose, 0.00-1.31 per cent; mannose, 0.80-1.16 per cent; crude fiber, 0.87-1.02 per cent; ashes, 1.40-4.40 per cent, and sodium chloride, 0.78-3.31 per cent. Mannose seems to form an important constituent of this dish, its quantity amounting to about 10 per cent of the dry solids.—*Zeitschr. f. Unters. d. Nahr. Genussm.*

International Photographic Exhibition, 1913.—It is proposed to hold a great photographic exhibition at Munich, Germany, from May the 15th to October the 15th, 1913. The exhibition will include artistic photography, reproduction technique, book printing and lithography, although the largest sections will be devoted to the photographic trade and industry and the graphic arts. The preparations will be placed under the control of Prof. Emmerich of the *Süddeutscher Photographen Verein E. V.*, who supervised the arrangements of the exhibition held in Munich in 1907 with so much success.—*Abels Photogr. Weekly.*

Trade Notes.

Fastening Oil Colors to Enameled Palettes.—A cement for this purpose is made as follows: Stir 2 parts of rye flour in 6 parts of hot water, then add 1 and let cool down. Finally add 2 parts per thousand pure carbolic acid.

Linoleum Cement.—Melt in a suitable vessel 70 grammes colophony, remove from the fire and stir until cooled off somewhat, then add 10 to 12 grammes castor oil and 13 to 16 grammes alcohol (90 per cent), stirring the while. Apply while still warm, or if kept in the dry state it must be heated first before using. The cement hardens quickly and has a wonderful adhesive power.—*Gummi Zeitung.*

Moving Pictures in Japan.—In Yokohama as well as throughout the empire the moving picture show is taking the place of the old-time theater. The price of admission is sometimes 2½ cents, but usually 5 cents. There are also numerous traveling picture shows. These concerns, according to a consular report, do a large film exchange business, mostly with European manufacturers. An American film is seldom seen, although investigation leads to the belief that the public would be very glad to see more characteristic American views.

Wood Required to Make One Ton of Paper.—The following estimate of the amount of wood required to make one ton of newspaper is given by H. S. Ferguson: One cord of wood will yield one ton of dry ground wood pulp. Two cords of wood will yield one ton of dry sulphite pulp. In the paper mill 22 per cent of the sulphite and 8 per cent of ground wood is wasted. One ton of paper containing 25 per cent of sulphite requires $25/98 \times 2 + 79/92 \times 1 = 1.32$ cords. One ton of paper containing 22½ per cent sulphite requires $22.5/98 \times 2 + 77.5/98 \times 1 = 1.30$ cords. One ton of paper containing 20 per cent sulphite requires $20/98 \times 2 + 80/98 \times 1 = 1.28$ cords.—*Paper.*

Paper Moldings of Monuments, Reliefs, etc.—The method of taking surface impressions or even true molds of monuments, sculptures, reliefs, inscriptions, etc., with the aid of wet paper seems to be but little known and practised. The following hints taken from *Zeitschrift für Ethnologie* may therefore be welcome and useful to the traveler: The paper most suitable for this purpose is copper-plate paper made of pure tough rags, although the paper used for wrapping oranges has also been recommended. The sheets are well soaked in water and separately laid on the stone which has been previously cleaned and wetted. The sheets are laid one on top of the other, and after each application the paper must be beaten with a brush handle or the like to force it into all the hollows. As this procedure causes a tearing of the paper, thus exposing parts of the stone, more sheets have to be used until the whole surface is covered. Then the whole is given an application of a good paste to be followed by more paper. For large pieces a second coating of size and paper is necessary, as the mold must be strong enough to retain its form after drying. For reliefs or shallow carved patterns a few sheets will suffice, however. Any cracks or cavities in the monument or wood carving which do not form any part of the object must be filled out with clay or soaked paper pulp before beginning the work. In hot weather the mold will be dry in about 24 hours, and it may then be pealed off carefully. During the night it should be kept covered to protect it from the morning dew. When entirely dry it should be given a coating of warm oil; dammara lac has also been recommended. A good paste for binding together the paper may be made in the cold by mixing together 250 grammes of rye flour and 750 grammes powdered and washed kaolin with water. In case the mold should break or tear it must at once be pasted together.

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