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# XII. On the two-dimensional motion of infinite liquid produced by the translation or rotation of a contained solid

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intensities at corresponding points have been made with reflecting surfaces of rectangular form or consisting of either two or three elements in the same plane for various angles of incidence; the results show that the expression for the illumination at any point of the diffraction pattern contains a factor proportional to the square of the cosine of the obliquity at such point.

The experiments and observations described in the note were carried out in the Palit Laboratory of Physics. The writer hopes to carry out further work on the subject of oblique diffraction by various forms of aperture, and particularly in regard to the positions of the points of maximum intensity in the pattern, which would no doubt differ from those given by the usual formulæ owing to the asymmetry of the illumination curves.

Calcutta,
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## XII. *On the Two-Dimensional Motion of Infinite Liquid produced by the Translation or Rotation of a Contained Solid.* By J. G. Leathem, *M.A.*, *D.Sc.*, *Fellow of St. John's College, Cambridge* *.

1. PERIODIC conformal transformations—that is, transformations by which doubly connected regions in the plane of a variable $z=x+iy$, externally unbounded and bounded internally by a closed polygon or curve, may be represented conformally and repeatedly upon successive semi-infinite strips of width $\lambda$ in the half-plane $\eta>0$ of a variable $\zeta=\xi+i\eta$—have been studied by the present writer in a previous paper †. It has there been shown how the knowledge of such a transformation for any particular curve makes possible the specification of a field of circulatory liquid flow (with or without logarithmic singularities) round a fixed solid body bounded by the curve.

It is now proposed to show that such knowledge makes possible also the determination of the field of irrotational motion due to any translation or rotation of the same solid in surrounding infinite liquid.

* Communicated by the Author.

† J. G. Leathem, "On Periodic Conformal Curve-Factors and Corner-Factors," Proc. Royal Irish Academy, vol. xxxiii. Sec. A, August 1916.

2. The geometrical, or $(z, \zeta)$, relation may be either in the form

$$z=f(\zeta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or in the differential form

$$dz = \mathscr{E}(\zeta)d\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\mathscr{E}(\zeta)$ is a periodic curve-factor of linear period $\lambda$ and angular period $2\pi$. With this angular period it is necessary * that both $\mathscr{E}(\zeta)$ and $f(\zeta)$ should, for $\eta$ great and positive, tend to infinity like $\exp(2\pi\eta/\lambda)$. In fact, $\mathscr{E}(\zeta)$ is expansible in the form

$$\mathscr{E}(\zeta) = \exp(-2\pi i\zeta/\lambda) \, . \, \Sigma c_s \exp(2\pi s i\zeta/\lambda), \quad . \quad (3)$$

where $s=0, 2, 3, 4 \ldots$, and the coefficients may be complex. The periodicity of $z$ makes it necessary † that $c_1=0$.

From integration of (3) it follows that

$$z=f(\zeta)=c+(i\lambda/2\pi)\exp(-2\pi i\zeta/\lambda) \, . \, \Sigma\{c_s/(1-s)\}\exp(2\pi s i\zeta/\lambda), \quad (4)$$

where $c$ is another complex constant ‡.

It is also to be noticed that, if $|\mathscr{E}(\zeta)| = h$, and if $c_s=\kappa_s \exp(i\gamma_s)$,

$$h^2=\exp\left(\frac{4\pi\eta}{\lambda}\right)\left[\kappa_0^2+\exp\left(-\frac{4\pi\eta}{\lambda}\right)2\kappa_0\kappa_2\cos\left(\frac{4\pi\xi}{\lambda}+\gamma_2-\gamma_0\right)\right.$$
$$+\exp\left(-\frac{6\pi\eta}{\lambda}\right)2\kappa_0\kappa_3\cos\left(\frac{6\pi\xi}{\lambda}+\gamma_3-\gamma_0\right)$$
$$\left.+\exp\left(-\frac{8\pi\eta}{\lambda}\right)\left\{2\kappa_0\kappa_4\cos\left(\frac{8\pi\xi}{\lambda}+\gamma_4-\gamma_0\right)+\kappa_2^2\right\}\ldots\right], \quad . \quad (5)$$

the terms containing ascending integral powers of

$$\exp(-2\pi\eta/\lambda).$$

* *L. c.* § 5.

† *L. c.* § 4.

‡ The problem of obtaining a transformation of the type of formula (4), so that a given closed curve shall correspond to $\eta=0$, is the same as that of the parametric representation of the given curve by a formula of the type

$$x+iy=m\exp(-i\phi)+\mu_0+\Sigma\mu \exp(is\phi),$$

where $\phi$ is a real parameter, $m$ and the $\mu$'s are complex constants, and $s$ takes positive integral values.

It is to be noted that a formula of this type need not represent a curve free from nodes unless the constants are suitably restricted.

It is understood that the boundary in the $z$ plane is the locus corresponding to $\eta=0$.

3. *Field of flow due to translation of the boundary.*—If the boundary have a velocity V in a direction making an angle $\mu$ with the axis of $x$, the superposition on the whole system of such uniform velocity as brings the boundary to rest gives an irrotational fluid motion which has zero normal velocity at the boundary and tends, for $z$ infinite, to flow V in the direction $\mu+\pi$. If this motion have velocity potential $\phi$ and stream-function $\psi$, so defined that the velocity is the upward gradient of $\phi$, and if $w\equiv\phi+i\psi$, $w$ must tend, for $z$ infinite, to the form

$$-\mathrm{V}z\exp(-i\mu)+\text{const.}, \quad . \quad . \quad . \quad . \quad (6)$$

or, in terms of $\zeta$,

$$-\mathrm{V}\kappa_0(i\lambda/2\pi)\exp i(\gamma_0-\mu-2\pi\zeta/\lambda). \quad . \quad . \quad (7)$$

Now if

$$w=-(\mathrm{V}\kappa_0\lambda/\pi)\sin(2\pi\zeta/\lambda-\gamma_0+\mu), \quad . \quad . \quad (8)$$

this tends, for $\eta\longrightarrow+\infty$, to the form (7); and as

$$\psi=-(\mathrm{V}\kappa_0\lambda/\pi)\cos(2\pi\xi/\lambda-\gamma_0+\mu)\sinh(2\pi\eta/\lambda),$$

it is clear that $\psi$ is zero along the boundary $\eta=0$. The corresponding form of $\phi$ shows that there is no circulation round the boundary; and $w$ is free from infinities in the relevant region.

Hence formula (8) specifies that irrotational motion past the fixed boundary whose limit form, at indefinitely great distance, is the assigned uniform flow.

4. *The impulse of the motion due to translation.*—Though modern speculation tends to regard wave-motion as the preponderating factor in suction and other inertia phenomena of floating bodies, it can hardly be doubted that, in the case of a submarine at least, the ordinary inertia coefficients measure approximately the resistance to quick changes of velocity. Thus the evaluation of the impulse (X, Y) of the combined motion of solid and fluid, when the solid has translatory motion, is of interest.

If an approximation to $w$ for $|z|$ great, closer than that afforded by formula (6), be

$$w=-\mathrm{V}z\exp(-i\mu)+\mathrm{C}+\mathrm{D}/z, \quad . \quad . \quad . \quad (6\,a)$$

where C and D are complex constants, it is known * that

$$X+iY=-2\pi\rho D,$$

where $\rho$ is the density of the liquid, and it is supposed that the mean density of the solid is also $\rho$.

If formula (6 $a$) be expressed in terms of $\zeta$ by means of formula (4), it yields the approximation

$$w=-V\kappa_0\left(\frac{i\lambda}{2\pi}\right)\exp i\left(\gamma_0-\mu-\frac{2\pi\zeta}{\lambda}\right)+C'$$

$$+\left\{V\kappa_2\left(\frac{i\lambda}{2\pi}\right)\exp i(\gamma_2-\mu)+D\left(\frac{2\pi}{i\lambda\kappa_0}\right)\exp(-i\gamma_0)\right\}\exp\left(\frac{2\pi i\zeta}{\lambda}\right)$$

where C′ is a constant. On comparison of this with formula (8), it appears that the coefficient of $\exp(2\pi i\zeta/\lambda)$ must equal $V\kappa_0(i\lambda/2\pi)\exp i(\mu-\gamma_0)$. Hence

$$D=-(V\lambda^2/4\pi^2)\{\kappa_0^2\exp(i\mu)-\kappa_0\kappa_2\exp i(\gamma_0+\gamma_2-\mu)\},$$

and therefore

$$X+iY=(\rho V\lambda^2/2\pi)\{|c_0|^2\exp(i\mu)-c_0c_2\exp(-i\mu)\}. \quad . \quad (9)$$

5. *Field of flow due to rotation of the boundary.*—When the boundary has a motion of rotation, the specification of the liquid motion presents greater difficulty; the outline of the procedure is as follows.

One motion is known which satisfies the proper condition at the moving boundary—namely, a rotation of the whole liquid, as if rigid, with the same angular velocity as the boundary. This may be called the *first motion*. It is not the required motion because it is rotational, and because it has infinite velocity at infinity.

Another motion, which will be called the *second motion*, can be specified. This also is rotational, having the same vorticity at every point as the first motion; but at the boundary its normal velocity is zero. It has infinite velocity at infinity.

If the second motion be subtracted from the first motion, the result is an irrotational motion whose normal velocity at the boundary is the same as that of the boundary itself. This may be called the *difference motion*. If the velocities of the first and second motions tend to equality at infinity in such manner that the difference motion tends to zero at infinity and has no circulation round the solid, the difference

* J. G. Leathem, "Some Applications of Conformal Transformation to Problems in Hydrodynamics," Phil. Trans. Roy. Soc., A, vol. ccxv. 1915, § 17.

motion satisfies all the requirements of the problem, and is the motion due to the rotation of the solid.

6. *The first motion.*—The first motion will be taken to be a rotation, as if rigid, with angular velocity $\omega$, about the point $z=\kappa \exp(i\gamma)$. There is no loss of generality in this choice, as the substitution of another centre of rotation can always be effected by superposing a motion due to translation of the boundary as explained in article 3.

The first motion may be specified by its stream function $\psi_1$, namely

$$\psi_1=-\tfrac{1}{2}\omega\ |\ z-\kappa \exp(i\gamma)\ |^2, \quad . \quad . \quad . \quad (10)$$

which is also expressible as a function of $\xi$ and $\eta$. Another specification is by $u_1$, $v_1$, where

$$u_1=\partial\psi_1/\partial\eta, \qquad v_1=-\partial\psi_1/\partial\xi, \quad . \quad . \quad (11)$$

these also being regarded as functions of $\xi$ and $\eta$. It is to be noted that $u_1$ and $v_1$ are not velocity components, but that the velocity components in the directions corresponding respectively to $\xi$ and $\eta$ increasing are $u_1/h$ and $v_1/h$.

From formula (4) it follows that

$$x=\kappa\cos\gamma+\frac{\lambda}{2\pi}\left\{\kappa_0 e^{2\pi\eta/\lambda}\sin\left(\frac{2\pi}{\lambda}\xi-\gamma_0\right)\right.$$
$$\left.+\sum_2\frac{\kappa_s}{s-1}e^{-2\pi(s-1)\eta/\lambda}\sin\left(\frac{2\pi(s-1)\xi}{\lambda}+\gamma_s\right)\right\},$$

$$y=\kappa\sin\gamma+\frac{\lambda}{2\pi}\left\{\kappa_0 e^{2\pi\eta/\lambda}\cos\left(\frac{2\pi}{\lambda}\xi-\gamma_0\right)\right.$$
$$\left.-\sum_2\frac{\kappa_s}{s-1}e^{-2\pi(s-1)\eta/\lambda}\cos\left(\frac{2\pi(s-1)\xi}{\lambda}+\gamma_s\right)\right\},$$

so that, when $\eta$ is great,

$$\psi_1=\frac{-\omega\lambda^2}{8\pi^2}\left\{\kappa_0^2 e^{4\pi\eta/\lambda}-2\kappa_0\kappa_2\cos\left(\frac{4\pi}{\lambda}\xi+\gamma_2-\gamma_0\right)\right.$$
$$\left.-\kappa_0\kappa_3 e^{-2\pi\eta/\lambda}\cos\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\ldots\right\}, \quad (12)$$

and

$$u_1=-\frac{\omega\lambda}{2\pi}\left\{\kappa_0^2 e^{4\pi\eta/\lambda}+\frac{1}{2}\kappa_0\kappa_3 e^{-2\pi\eta/\lambda}\cos\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\ldots\right\},$$
$$. \quad . \quad . \quad . \quad . \quad (13)$$

$$v_1=\frac{\omega\lambda}{2\pi}\left\{2\kappa_0\kappa_2\sin\left(\frac{4\pi}{\lambda}\xi+\gamma_2-\gamma_0\right)\right.$$
$$\left.+\frac{3}{2}\kappa_0\kappa_3 e^{-2\pi\eta/\lambda}\sin\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\ldots\right\}, \quad (14)$$

the terms being arranged in descending order of importance.

7. *The second motion.*—The second motion, a rotational motion having zero normal velocity at the boundary, is got by an imaging of the vortex distribution; and the utility of the periodic conformal transformation consists in the fact that it makes this imaging process possible.

The specification of the motion may be by a stream-function $\psi_2$, or by functions $(u_2, v_2)$ equal to $\partial\psi_2/d\eta$, $-\partial\psi_2/\partial\xi$, the corresponding velocity components in the $z$ plane being $u_2/h$, $v_2/h$.

It is convenient, for a moment, to think of $\psi_2$ as the stream-function of a motion in the $\zeta$ plane, of which $(u_2, v_2)$ would be the velocity. If such a motion, unhampered by any rigid boundary, were due to a line-vortex representing a circulation $m$, situated at $\zeta=\zeta'$, and periodically repeated at $\zeta'\pm r\lambda$, $(r=1, 2, 3, \ldots)$, it is known * that the corresponding stream-function would be

$$-\frac{m}{2\pi}\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\right|.$$

When the line $\eta=0$ is a rigid boundary an image must be introduced at the point $\zeta=\zeta''$, where $\zeta''$ is the complex conjugate to $\zeta'$, and the stream-function is

$$-\frac{m}{2\pi}\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\Big/\sin\frac{\pi}{\lambda}(\zeta-\zeta'')\right|.$$

Instead of single vortices a continuous periodic distribution of vorticity may be postulated over the whole area between $\eta=0$ and $\eta=t$, the former line being still a rigid boundary; and if $m$ be replaced by $\sigma(\zeta')dS'$, where $dS'$ is an element of area and $\sigma$ a density of distribution, the stream-function is

$$-\frac{1}{2\pi}\int\sigma(\zeta')\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\Big/\sin\frac{\pi}{\lambda}(\zeta-\zeta'')\right|dS', \quad (15)$$

the area integral being taken over a rectangle of length $t$ and breadth $\lambda$.

If $dS$ in the $z$ plane and $dS'$ in the $\zeta$ plane be corresponding elements of area,

$$dS = \{h(\zeta')\}^2 dS';$$

so, if the circulation round the contour of any area in the $\zeta$ plane is to be equal to $2\omega$ times the corresponding area in the $z$ plane, it is necessary that

$$\sigma(\zeta') = 2\omega\{h(\zeta')\}^2. \quad . \quad . \quad . \quad . \quad (16)$$

* Proc. Royal Irish Academy, *l. c.* § 13.

The stream-function is then

$$\psi_t = -\frac{\omega}{\pi}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \log \left| \frac{\sin\frac{\pi}{\lambda}(\zeta-\zeta')}{\sin\frac{\pi}{\lambda}(\zeta-\zeta'')} \right| d\xi'\,d\eta', \qquad (17)$$

and this specifies a motion in the $z$ plane which has vorticity $\omega$ at all points for which $t > \eta > 0$, and has the curve corresponding to $\eta=0$ as a fixed boundary.

The corresponding $u$, $v$ functions are

$$u_t = \frac{\partial\psi_t}{\partial\eta} = -\frac{\omega}{\lambda}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \left\{ \frac{\sinh\frac{2\pi}{\lambda}(\eta-\eta')}{\cosh\frac{2\pi}{\lambda}(\eta-\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} - \frac{\sinh\frac{2\pi}{\lambda}(\eta+\eta')}{\cosh\frac{2\pi}{\lambda}(\eta+\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} \right\} d\xi'\,d\eta', \quad (18)$$

$$v_t = -\frac{\partial\psi_t}{\partial\xi} = \frac{\omega}{\lambda}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \left\{ \frac{\sinh\frac{2\pi}{\lambda}(\xi-\xi')}{\cosh\frac{2\pi}{\lambda}(\eta-\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} - \frac{\sin\frac{2\pi}{\lambda}(\xi-\xi')}{\cosh\frac{2\pi}{\lambda}(\eta+\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} \right\} d\xi'\,d\eta'. \quad (19)$$

It is to be noticed that if $\zeta$ is inside the area of integration the subjects of integration in (17), (18), and (19) have infinities at $\zeta'=\zeta$; these infinities, however, are not sufficiently powerful to make the integrals divergent.

8. These formulæ may be checked by noting directly what conditions $u_t$, $v_t$ must satisfy if they are to represent the kind of motion in the $z$ plane which has been described in the previous article.

$\xi$ and $\eta$ are curvilinear coordinates in the $z$ plane, and the corresponding velocity components are $U=u_t/h$ and $V=v_t/h$. The boundary condition $V=0$ or $v_t=0$, when $\eta=0$, is clearly satisfied. The equations of continuity and of vorticity are respectively

$$\frac{\partial}{\partial \xi}(\mathrm{U}h)+\frac{\partial}{\partial \eta}(\mathrm{V}h)=0, \quad \frac{\partial}{\partial \xi}(\mathrm{V}h)-\frac{\partial}{\partial \eta}(\mathrm{U}h)=2\omega h^2,$$

or

$$\frac{\partial u_t}{\partial \xi}+\frac{\partial v_t}{\partial \eta}=0, \quad \frac{\partial v_t}{\partial \xi}-\frac{\partial u_t}{\partial \eta}=2\omega\{h(\zeta)\}^2, \quad . \quad (20)$$

when $\zeta$ is inside the area of integration.

The testing of these equalities involves the differentiation of the integrals of formulæ (18) and (19), and this cannot be done by the ordinary rule of differentiation under the sign of integration, since that would yield semi-convergent integrals. It is, however, easy to apply the method of differentiation explained in the Cambridge Tract on 'Volume and Surface Integrals used in Physics,' articles 21 and 23, and it is then readily verified that $u_t$ and $v_t$ satisfy both conditions.

A single compact formula giving both $u_t$ and $v_t$ is

$$v_t+iu_t=\frac{\omega}{\lambda}\int_0^t\int_0^\lambda\{h(\zeta')\}^2\left[\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')\right]d\xi'\,d\eta'. \quad (21)$$

The integral on the right-hand side has the appearance of being a function of the complex variable $\zeta$; but this appearance is deceptive, for if $v$ and $u$ were conjugate functions there would be no vorticity *.

9. The next step that suggests itself is a passage to limits for $t$ infinitely great. This is feasible in the case of $v_t$, but the integral representing $u_t$ proves to be divergent.

It will be shown that this can be remedied by adding to $u_t$, before passage to limit, a suitable function of $t$ which does not involve $\xi$ or $\eta$. An addition to $u_t$ means simply the superposition of an irrotational motion with circulation round the fixed boundary. This will serve to cancel an undesired circulation at infinity in the motion defined by $u_t$ and $v_t$.

No corresponding addition to $v_t$ need be or could be made.

It being necessary to consider not only the convergence of the $u$ and $v$ integrals but also the forms to which these, regarded as functions of $\xi$ and $\eta$, tend for $\eta$ very great and positive, it is important to notice two expansions of the function which appears in square brackets in formula (21).

* On this point compare the writer's note "On Functionality of a Complex Variable" in the 'Mathematical Gazette,' early in 1918.

If $\eta' > \eta > 0$

$$\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')$$
$$= 4i\left[\tfrac{1}{2}+\Sigma\exp\left(\frac{-2s\pi\eta'}{\lambda}\right)\cosh\frac{2s\pi}{\lambda}\{\eta-i(\xi-\xi')\}\right], \quad (22)$$

and if $\eta > \eta' > 0$

$$\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')$$
$$= -4i\Sigma\sinh\left(\frac{2s\pi\eta'}{\lambda}\right)\exp\frac{2s\pi}{\lambda}\{i(\xi-\xi')-\eta\}, \quad (23)$$

where $s=1, 2, 3, \ldots$. The formulæ can be verified by noticing that each side of each equality is equivalent to

$$2i\left[\frac{1}{1-\exp\frac{2\pi}{\lambda}\{\eta-\eta'-i(\xi-\xi')\}}-\frac{1}{1-\exp\frac{2\pi}{\lambda}\{\eta+\eta'-i(\xi-\xi')\}}\right].$$

For $\eta'$ great the most important terms of $\{h(\zeta')\}^2$ are

$$\{h(\zeta')\}^2 = \kappa_0^2\exp\left(\frac{4\pi\eta'}{\lambda}\right)+2\kappa_0\kappa_2\cos\left(\frac{4\pi\xi'}{\lambda}+\gamma_2-\gamma_0\right)$$
$$+2\kappa_0\kappa_3\exp\left(-\frac{2\pi\eta'}{\lambda}\right)\cos\left(\frac{6\pi\xi'}{\lambda}+\gamma_3-\gamma_0\right)+\ldots, \quad (24)$$

the terms decreasing by successive negative powers of $\exp(2\pi\eta'/\lambda)$; it is to be noted that the functions of $\xi'$ which multiply these exponentials are sums which may contain constant as well as harmonic terms.

In studying the form of the subject of integration in formula (21), for great values of $\eta'$, with a view to examining the divergence for $t\to\infty$, the product of the series (22) and (24) may be used. For this purpose a term of the product may be ignored if its integral with respect to $\xi'$ through a range $\lambda$ is zero, or if it contains as factor the exponential of a negative multiple of $\eta'/\lambda$. By one or other of these tests every term is negligible except one, namely,

$$2i\kappa_0^2\exp(4\pi\eta'/\lambda),$$

which contributes to the integral, at its upper limit (after integration with respect to $\xi'$),

$$(i\kappa_0^2\lambda^2/2\pi)\exp(4\pi t/\lambda).$$

Hence the subtraction of $(i\kappa_0{}^2\omega\lambda/2\pi)\exp(4\pi t/\lambda)$ from the right-hand side of formula (21) gives an expression which has a definite limit for $t\rightarrow\infty$. This justifies the definition

$$v_2+iu_2=\frac{\omega}{\lambda}\operatorname*{Lim}_{t\rightarrow\infty}\left[-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}\right.$$
$$\left.+\int_0^t\int_0^\lambda\{h(\zeta')\}^2\left\{\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')\right\}d\xi'\,d\eta'\right]. \quad (25)$$

10. *Limiting form of the second motion at infinity.*—The formula (25) defines a motion which has all the characteristics required for the second motion, with the, as yet, possible exception of tending to the proper form at infinity. It is now necessary to inquire what are the limiting forms to which $u_2$ and $v_2$, functions of $\xi$ and $\eta$, tend with indefinite increase of $\eta$.

For this purpose the series-expansions of formulæ (22) and (23) may be used, each within the appropriate range of $\eta'$. If [22], [23] be used as abbreviations for the expressions on the right-hand side of these formulæ,

$$v_2+iu_2=\frac{\omega}{\lambda}\operatorname*{Lim}_{t\rightarrow\infty}\left[-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}+\int_\eta^t\int_0^\lambda\{h(\zeta')\}^2[22]\,d\xi'\,d\eta'\right.$$
$$\left.+\int_0^\eta\int_0^\lambda\{h(\zeta')\}^2[23]\,d\xi'\,d\eta'\right]. \quad (26)$$

As the integral of formula (25) is absolutely convergent in respect of the infinity of the subject of integration at $\zeta'=\zeta$, it is safe to use the series [22] and [23] right up to the critical value $\eta'=\eta$ which separates the ranges within which they are respectively valid. For $\{h(\zeta')\}^2$ the series of formula (24) is again employed.

In taking the term-by-term products of the two series which are multiplied under the sign of integration, any resulting term may be passed over whose integral with respect to $\xi'$ over a range $\lambda$ is zero. Thus a term of the the type

$$\cos\{(2\pi m/\lambda)(\xi'+\alpha)\}\cos\{(2\pi n/\lambda)(\xi'+\beta)\}$$

need not be considered unless $m=n$. Further, when only an approximation for $\eta$ great is desired, an estimate of the importance of an exponential in $\eta'$ and $\eta$ is to be made on the hypothesis that $\eta$ is very great but that $\eta'$ is of a higher order of greatness at one of the limits of integration, while $\eta'=\eta$ at another of the limits. These considerations reduce the important terms in the equivalent of the square bracket in formula (26) to

$$-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}+4i\int_\eta^t\int_0^\lambda\Big[\frac{1}{2}\kappa_0{}^2e^{4\pi\eta'/\lambda}$$

$$+2e^{-4\pi\eta'/\lambda}\kappa_0\kappa_2\cosh\frac{4\pi}{\lambda}\{\eta-i(\xi-\xi')\}\cos\left(\frac{4\pi}{\lambda}\xi'+\gamma_2-\gamma_0\right)\Big]d\xi'd\eta'$$

$$-4i\int_0^\eta\int_0^\lambda 2\kappa_0\kappa_2\sinh\left(\frac{4\pi\eta'}{\lambda}\right)\cos\left(\frac{4\pi}{\lambda}\xi'+\gamma_2-\gamma_0\right)$$

$$\times\exp\frac{4\pi}{\lambda}\{i(\xi-\xi')-\eta\}\,d\xi'\,d\eta',$$

which reduces, after omission of some negligible elements, to a form whose limit, for $t\rightarrow\infty$, on substitution in (26) yields the formula

$$v_2+iu_2\backsim\frac{\omega\lambda}{2\pi}\left[-i\kappa_0{}^2\exp\left(\frac{4\pi\eta}{\lambda}\right)+2\kappa_0\kappa_2\sin\left(\frac{4\pi\xi}{\lambda}+\gamma_2-\gamma_0\right)\right]. \quad (27)$$

11. *The difference motion.*—If formula (27) be compared with formulæ (13) and (14) it is seen that

$$v_2+iu_2-(v_1+iu_1)\rightarrow 0. \quad (28)$$

Hence if $(u, v)$ specify the difference motion, so that

$$u=u_1-u_2,\quad v=v_1-v_2,$$

$u$ and $v$ tend to zero at infinity.

Thus the difference motion is an irrotational motion, vanishing at infinity and having at the boundary a normal velocity corresponding to rotation about the point $\kappa\exp(i\gamma)$. It is free from circulation, as a circulation would involve, for $\eta$ infinite, a definite limit value of $u$ different from zero. It therefore constitutes the solution of the problem of motion due to the rotation of the boundary.

12. *Forms of boundary to which the method applies.*—The applicability of this method to solving the problems of motion due to translation and rotation depends upon the knowledge of a periodic conformal transformation which will make any particular form of boundary correspond to the real axis in the $\zeta$ plane. That a considerable variety of such transformations and their corresponding boundaries is

available is demonstrated in the writer's paper on the subject referred to above. In particular, mention may be made of polygonal boundaries (*l. c.* § 8); and it may be noticed that for a regular polygon of $n$ sides the transformation is

$$\frac{dz}{d\zeta} = \mathrm{K}\left[\prod_{s=0}^{s=n-1} \sin\frac{\pi}{\lambda}\left(\zeta-\frac{s\lambda}{n}\right)\right]^{\frac{2}{n}} = \frac{\mathrm{K}}{4}\left\{-2\sin\frac{n\pi\zeta}{\lambda}\right\}^{\frac{2}{n}}, \quad (29)$$

so that

$$h^2 = \frac{\mathrm{K}^2}{16}\left[2\left\{\cosh\frac{2n\pi\eta}{\lambda} - \cos\frac{2n\pi\xi}{\lambda}\right\}\right]^{\frac{2}{n}}, \quad . \quad . \quad (30)$$

K being a constant; the latter expression, with accented letters, would be the first factor under the sign of integration in formula (25).

In all cases where the periodic transformation is known the solution of the hydrodynamical problems is reduced to quadratures.

In certain cases the integrations can be completed; this is noticeably the case when $f(\zeta)$, and therefore also $h^2$, is the sum of a finite number of terms harmonic in $\zeta$. The integration may be accurately effected by the method used for approximation in article 10 above. Of the terms arising from the multiplication of $h^2$ into the series [22] and [23] there are only a finite number which do not yield zero result when integrated with respect to $\xi'$ through a range $\lambda$, and each of these can be integrated separately with respect to $\eta'$.

The simplest example is the ellipse, for which the transformation is

$$z = c\cosh\{\alpha-(2\pi i/\lambda)\zeta\}, \quad . \quad . \quad . \quad . \quad (31)$$

so that

$$h^2 = \frac{2\pi^2c^2}{\lambda^2}\left\{\cosh 2\left(\alpha+\frac{2\pi}{\lambda}\eta\right) - \cos\left(\frac{4\pi}{\lambda}\xi\right)\right\}. \quad . \quad (32)$$

The working out of this case may be used to test the method, as the results are otherwise known.

Another simple integrable case corresponds to a boundary whose polar equation is

$$r = a+2b\cos 2\theta, \quad (a>2b). \quad . \quad . \quad . \quad (33)$$

The transformation is

$$z = b\exp(-2\pi i\zeta/\lambda) + a\exp(2\pi i\zeta/\lambda) + b\exp(6\pi i\zeta/\lambda). \quad (34)$$

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