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ON THE THEORY OF GROUPS OF FINITE ORDER

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(*Presidential Address.*)

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It has been suggested to me that I should take advantage of the present occasion to give an account of the recent progress of the theory of groups of finite order. That very considerable advance has been made in the last twenty, and especially in the last ten years, is undoubtedly the case. That advance, however, has been in a great variety of directions, and it is probably too soon, as yet, to present its different parts in their proper proportion and perspective. It would not, I think, be possible to do at the present time for the theory of groups of finite order what was done so ably for the allied theory of algebraic numbers by Prof. Hilbert in his report of 1897.

But a more serious objection to any attempt on my part to give, on the present occasion, an account of the recent advance in the theory is that such an account would certainly be uninteresting to a considerable number of my audience. It is undoubtedly the fact that the theory of groups of finite order has failed, so far, to arouse the interest of any but a very small number of English mathematicians; and this want of interest in England, as compared with the amount of attention devoted to the subject both on the Continent and in America, appears to me very remarkable. I propose to devote my address to a consideration of the marked difference in the amount of attention devoted to the subject here and elsewhere, and to some attempt to account for this difference.

It is to our younger students that one would naturally look for a shew of interest in a practically new subject, for such the theory of groups in its modern developments actually is. Unfortunately it is the case that so far one has looked almost in vain. The subject, under the head "Theory of Equations, including the Theory of Substitutions," has formed part of the schedule for Part II. of the Mathematical Tripos since 1887. In the twenty-one years during which these regulations with modifications have been in force, questions directly on the theory of groups of finite order have been proposed on four occasions only; and while this shews that except on those four occasions no candidates have taken up the subject, anyone who has had experience in examining in the Mathematical Tripos will know that there have not necessarily been even four candidates who have made a serious study of the subject. At Oxford questions on the subject have appeared on one or two occasions in the examination for the Senior Mathematical Scholarship. The evidence of examinations, then, so far as it goes, is clearly that our younger mathematicians do not study the subject. As regards teaching, the evidence is even more pronounced. So far as I have been able to learn no course of lectures has ever been delivered either at Oxford or at Cambridge on the theory of groups of finite order. Moreover, in courses on the theory of algebraic equations, the amount of group-theory introduced is very small, as also is the number of students attending such lectures. In fact, so far as the teaching of the subject in England is concerned, one may say that it does not exist. The following facts, which I owe to the kindness of correspondents, will shew that abroad and in America the state of things is very different.

In Paris, M. Jordan gives a course on the theory of groups of finite order at the Collège de France at regular intervals to an average class of six students, while the Galois theory of equations is lectured on at the Sorbonne and the Ecole Normale, as well as at one or two of the provincial Universities.

In most German Universities the regular course of lectures on algebra, attended by large classes of students, contains an exposition of the more elementary parts of the theory of groups of permutations. In addition to this there are, in all the larger Universities, special courses devoted to groups of finite order, and to discontinuous groups, which attract a considerable number of students. For instance, such special courses last year were attended at Göttingen by thirty students, and at Freiburg by twelve.

In the United States all the leading Universities offer regular courses in the theory of groups of finite order, with the exception of Harvard, where a course is given on the Galois theory of equations. In some cases the course is a yearly one, and in others it is biennial. These courses attract from two or three up to ten or twelve students, who, in general, have already taken the B.A. degree.

As regards the teaching of the subject, then, the contrast between the state of things in this country and abroad is sufficiently striking. Though it may possibly be the case that elsewhere the demand for the teaching has been to some extent created by a persistent supply, there can be no doubt that, if even a quite small number of our senior mathematical students at the Universities wished for courses of lectures on group-theory, the supply would be forthcoming. Can any adequate reason be given, then, to account for the entire lack of any such demand? The objection that the subject has at present no obvious applications to matters of practical interest is one which may be, and has been, made to other branches of pure mathematics—for instance, to the theory of forms—which nevertheless attract a considerable number of students. The abstract nature of the subject, and the difficulty of seeing at first what it can lead to, and how it is capable of any great extent of development, have also been suggested as deterrents to a beginner; but these points again are not found to prevent students from becoming rapidly interested in other branches of pure mathematics. Moreover, the theory of groups of finite order has, from a beginner's point of view, one distinct matter in its favour, viz., that there is no complicated notation to be learnt, that used being almost self-explanatory from the start. Why, then, again the almost total neglect of the subject?

In my belief it is, to a considerable extent, due to the fact that, in the teaching of junior students, opportunities which occur for familiarising them with the ideas that are fundamental in group-theory, when those ideas present themselves in simple form, are not taken advantage of. It is the results, and not the processes which lead to the results, on which their attention is mainly fixed; *e.g.*, the product, and not the act of multiplication, when once the latter has ceased to be a vexation. So long as the processes are arithmetical or algebraical, the separation of the process from its operand, and the point of view from which the process is regarded as a definite operation that can be effected on any operand, is one which the young mind grasps with difficulty, if at all. But in the more concrete field of geometry this is, I think, not so. Thus, to take a very simple example, when inverse points with respect to a circle have been defined, the operation of inversion with respect to a circle, regarded as a process for passing from any given plane figure to its inverse, is one which the beginner should grasp at once, aided, if necessary, by the fact that there is an instrument (a Peaucillier's cell), by the use of which the operation can always be effected ; and the statement that inversion with respect to a circle is an operation of order two should have a perfectly definite and simple geometrical meaning for him. But of the students who will use the process of inversion with skill in dealing with some particular geometrical problem, how many ever form this idea of the process and learn to regard it apart from the special result they have to prove? So with plane kinematics, translations and rotations should be presented as operations which can be carried out on any plane figure, quite independently of its size and position ; and the student would thus obtain further conceptions of operations of finite order, and of the distinction between those of finite and those not of finite order. Moreover, without going beyond the powers of a junior student, the idea of a cyclical group arising from a rotation of finite order, and of the dihedral group on which the theory of the kaleidoscope depends, might well be dealt with. The point I wish to emphasize is that, by attending to the actual processes, as well as to the results of the various elementary geometrical constructions that have to be presented to a student, he would be led to acquire quite naturally, at an early stage, ideas and points of view which at present he in general misses; and for want of which, later on, group-theory may entirely fail to appeal to him.

At a somewhat later stage, when the student has passed from elementary to projective geometry, a projective transformation, either of the plane or of space, should be presented to him (it hardly ever is) as a definite operation to be performed on any geometrical figure, the process being independent of the figure. As a further illustration of what might be done at this stage, the set of twenty-four collineations which transform a quadrangle into itself might be considered. They form a group which is, of course, abstractly equivalent to that of the permutations of four symbols. But while in the latter form the problem presented would to many minds be almost repulsive in its naked formality, the study of the quadrangle, from the point of view of the set of collineations for which it is invariant, could hardly fail to appeal to a mind with a geometrical bent. At each stage in his progress, especially on the geometrical side, the student whose preparation has been suitable would find, without going aside from the beaten track, further problems in which geometrical and group-theory interests are combined.

The remarks that have been made are not intended to suggest that the details of group-theory are suitable subjects for teaching in the earlier parts of a student's course. Neither is the theory of forms. But just as from a quite early stage, some of the fundamental ideas of the latter, *e.g.*, the idea of invariance, is introduced ; so should be the fundamental ideas of the former, and, when the opportunity offers, these should be illustrated by simple applications. Were this done systematically, the student's outlook would become wider than it generally is at present. He would gradually recognize that all the processes he uses may be regarded, each set in its own connection, as combining together according to definite laws; in fact, he would naturally, as a necessary result of his training, be led to what can only be described as the group-theory point of view. As things are at present a student, who approaches the study of group-theory, has in general had no such preliminary training. His mind is not already stored with a number of particular cases which lead up to interest in a general theory, and it is probable that he is often, in consequence, repelled from the subject by a too early presentment of the purely formal side of the theory.

The statement that from three symbols E, A, B obeying the associative law of multiplication, and the relations

$$E^2 = A^2 = B^3 = (AB)^5 = E,$$

$$EA = AE = A, \qquad EB = BE = B,$$

just sixty distinct products can be formed (this is a formal statement involving all the properties of the icosahedral group), looks on the face of it as affording a merely curious illustration of non-commutative multiplication, while a proposal to verify the statement appears equivalent to proposing a series of conundrums. There would be nothing here to attract the student or to suggest anything but the driest formalism utterly divorced from any of his previous mathematical studies. Now the treatises which deal directly with the theory of groups of finite order, all begin more or less from the formal side. It is not, of course, the case that the student is confronted immediately with statements like the one just made. Nevertheless he has to begin by grasping a whole series of new ideas, such as group, sub-group, conjugate set, factor-group, isomorphism, presented to him in a purely abstract form; and the concrete illustrations which will connect these new ideas, in some directions, with matters with which he is already familiar come later.

No one has yet attempted to do for the student of this subject what has been done so brilliantly by Sir George Greenhill for the student of elliptic functions. Sir George Greenhill begins with the pendulum, presenting to the reader the one physical problem which a student of pure mathematics is almost sure to know something about; and he says in effect, if not in so many words—the horizontal distance of the bob of a pendulum from the vertical through the point of support is an elliptic function of the time. The reader has before him immediately a concrete example of an elliptic function, connected with more or less familiar ideas, to excite his interest and to draw him on. Sooner or later, if he is a pure mathematician, he will be led to the doubly connected Riemann's surface, the integrals upon it, and the class of algebraic functions connected with it.

It is true, as I have already stated, that no one has done for the various applications of groups of finite order what Sir George Greenhill has done for the applications of elliptic functions. Still, in this connection, a book must be referred to which offers probably the most perfect example we have of the application of a mathematical theory to a particular problem. That book is Prof. Klein's *Vorlesungen über das Icosaeder und die Auflösung der Gleichungen vom fünften Grade.* It was published twenty-four years ago, and for that reason perhaps is not very well known to our younger mathematicians. It forms the first of a series of works dealing with the general theory of automorphic functions; but, as it is exclusively concerned with the case in which the functions are algebraic, it is self-contained and makes no excessive demand for previous knowledge on the part of the reader. The latter part of the book gives a beautiful and masterly treatment of the equation of the fifth degree by means of the icosahedral function; but what I wish specially to draw attention to is that the earlier part forms an ideal introduction to the theory of groups of finite order.

The one kind of geometrical operation which can be dealt with directly and simply in an experimental way is the displacement of a solid body. One can take a solid body in one's hands and determine by actual experiment the various displacements of which it is capable without altering the position of its bounding surface. The groups of rotations of the regular solids give therefore the most concrete illustrations of groups of finite order. They are amenable to direct experimental treatment. It is these with which Prof. Klein deals. They are first considered from the geometrical, or experimental, point of view. Then the groups of linear transformations which are equivalent to them, which are, in fact, their translation into analysis, are introduced. Lastly, the functions which are invariant for these groups of linear transformations are determined, and the relations between them obtained. The reader is led naturally from the concrete to the abstract, and acquires by actual instances the new ideas necessary to the theory. He is shewn from the beginning the various lines of interest that open out and the points in which the theory is naturally connected with other branches of pure mathematics. His attention is not distracted by the purely formal developments of the theory which necessarily constitute so large a part of any treatise on the theory of finite groups in general.

The latter part of Prof. Klein's book is devoted to a special problem, viz., the solution of the quintic equation. This may or may not appeal to any given pure mathematician. I am sure, however, that such general knowledge of the theory of groups of finite order as is to be gained by a study of the first part, ought to form part of the equipment of every pure mathematician; and, until the subject is included among those lectured on at the Universities, I know of no better book from which to obtain such general knowledge than that of Prof. Klein.

I wish, in conclusion, to appeal to those who have the teaching of our younger pure mathematicians to do something to stimulate the study of group-theory in this country. If, when advice is given for the course of study to be pursued, the importance of some knowledge of group-theory for a pure mathematician (which is generally recognized elsewhere) were insisted on, there is little doubt but that a demand for the serious teaching of the subject would soon arise.